1 Klein-Gordon equation

Klein-Gordon (K.G.) equation is a relativistic wave equation for spin zero particles. $\Rightarrow$ The wave function has only 1 component: $\psi(x)$, which must be Lorentz invariant: $\psi'(x') = \psi(x)$.

To get such a wave equation, we square the Dirac equation:

$$\begin{aligned} i\hbar\dot{\psi} &= H\,\psi \Rightarrow -\hbar^2\ddot{\psi} = H^2\,\psi = \left(-\hbar^2 c^2\,\Delta + m^2c^4\right)\psi \\ &\Rightarrow \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + \left(\frac{mc}{\hbar}\right)^2\psi = 0 \end{aligned}$$

This give the K.G. equation in the form

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\psi(x) = 0 \tag{1.1}$$

where the d'Alembert operator is defined by (see No. 1) $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$. Plane wave solutions of (1.1) are of the form

$$\psi_{\vec{p}}(\vec{x},t) = N\,e^{-i(Et-\vec{p}\cdot\vec{x})/\hbar} \tag{1.2}$$

They are eigenfunctions of the momentum operator $\hat{\vec{p}} = -i\hbar\vec{\nabla}$ with eigenvalue $\vec{p}$, and $N(p)$ is a normalization constant. In order that (1.2) is a solution of (1.1), E must have the form

$$E^2 = \vec{p}^2c^2 + (mc^2)^2 \Rightarrow E = \pm\sqrt{\vec{p}^2c^2 + (mc^2)^2} \equiv \pm E_p \tag{1.3}$$

where $E_p = \sqrt{\vec{p}^2c^2 + (mc^2)^2} > 0$.

Does this mean negative energy? No ! For the Klein-Gordon case, E is not the eigenvalue of some Hamiltonian, but just the "frequency" of the solutions (1.2): $E = E_p > 0$ means positive frequency, and $E = -E_p < 0$ means negative frequency:

$$\psi_{\vec{p}}^{(+)}(\vec{x},t) = N(p)\,e^{-i(E_pt-\vec{p}\cdot\vec{x})/\hbar} \tag{1.4}$$

$$\psi_{\vec{p}}^{(-)}(\vec{x},t) = N(p)\,e^{-i(-E_pt-\vec{p}\cdot\vec{x})/\hbar} \tag{1.5}$$

We will show later that for both cases the energy is positive.

Current conservation

Multiplying the K.G. equation (1.1) by ψ^*, and the c.c. of (1.1) by ψ, and taking the difference of these two equations, we obtain

$$\partial_\mu\left[\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\right] = 0 \tag{1.6}$$

This has the form of current conservation: $\partial_\mu j^\mu = 0$. However, we cannot interpret j^0 as a "probability density", because it is <u>not positive definite</u>!

If we multiply the current in Eq.(1.6) by $i\hbar q$, where $q > 0$ is the electric charge of the particle, we obtain $\partial_\mu j_c^\mu = 0$, where

$$j_c^\mu = i\hbar q \left[\psi^* \partial^\mu \psi - \psi\, \partial^\mu \psi^*\right] \tag{1.7}$$

We can interpret $j_c^\mu = (c\,\rho_c, \vec{j}_c)$ as the "electric 4-vector current": The "charge density" is given by

$$\rho_c = \frac{i\hbar}{c^2} q \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \tag{1.8}$$

If we insert the solutions (1.4) and (1.5) into (1.8), we obtain

$$\begin{aligned} \rho_c^{(+)} &= \frac{i\hbar}{c^2} q \left(\frac{-2iE_p}{\hbar} \right) N(p)^2 = \frac{2E_p q}{c^2}\, N^2 \equiv \frac{q}{V} \\ \rho_c^{(-)} &= \frac{i\hbar}{c^2} q \left(\frac{2iE_p}{\hbar} \right) N(p)^2 = -\frac{2E_p q}{c^2}\, N^2 \equiv \frac{-q}{V} \end{aligned}$$

where we have set the normalization factor equal to

$$N(p) = \sqrt{\frac{c^2}{2E_p V}} \tag{1.9}$$

Therefore the solution (1.4) describes a <u>particle</u> with charge $q > 0$, and (1.5) describes the <u>antiparticle</u> with charge $-q < 0$. Therefore we can interpret the conserved current (1.7) as the electric current [1].

<u>Home work:</u> Use the "minimal substitution" (see No. 7) $\partial^\mu \to \partial^\mu + \frac{iq}{\hbar c} A^\mu$ to obtain the Klein-Gordon equation in an external electromagnetic field A^μ, and derive the current conservation for this case. Show that the conserved electric current is then given by

$$j_c^\mu = i\hbar q \left[\psi^* \partial^\mu \psi - \psi\, \partial^\mu \psi^* + \frac{2iq}{\hbar c} A^\mu\, \psi^*\, \psi \right]$$

Show that this current is invariant under the local gauge transformations given in No. 7.

[1] In order to describe also neutral particle consistently with the Klein-Gordon equation, one needs the methods of quantum field theory.

Lagrangian and Hamiltonian for Klein-Gordon field

The Lagrangian density for the free Klein-Gordon field is given by

$$\frac{1}{\hbar^2}\,\mathcal{L} = (\partial_\mu \psi^*)\,(\partial^\mu \psi) - \left(\frac{mc}{\hbar}\right)^2\,\psi^*\,\psi \tag{1.10}$$

Check this: The requirement that $\delta S = 0$ under variations of the fields ψ and ψ^* (and their derivatives) gives the Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} &= 0 \\
\frac{\partial \mathcal{L}}{\partial \psi^*} - \partial_\mu\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} &= 0
\end{aligned}$$

For the Lagrangian density (1.10), these equations become identical to the Klein-Gordon equations for ψ and ψ^*.

For the transformation to the Hamiltonian density, we need the "canonical momenta" of ψ and ψ^*:

$$\begin{aligned}
\Pi_\psi &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\hbar^2}{c^2}\,\dot{\psi}^* \equiv \Pi \\
\Pi_{\psi^*} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = \frac{\hbar^2}{c^2}\,\dot{\psi} = \Pi^*
\end{aligned}$$

Then the Hamiltonian density is given by

$$\begin{aligned}
\mathcal{H} &= \Pi\,\dot{\psi} + \Pi^*\,\dot{\psi}^* - \mathcal{L} = 2\left(\frac{c^2}{\hbar^2}\right)\Pi\,\Pi^* - \left(\frac{c^2}{\hbar^2}\right)\Pi\,\Pi^* + \hbar^2\left(\vec{\nabla}\psi^*\right)\cdot\left(\vec{\nabla}\psi\right) + (mc)^2\,\psi^*\,\psi \\
&= \left(\frac{c^2}{\hbar^2}\right)|\Pi|^2 + \hbar^2|\vec{\nabla}\psi|^2 + (mc)^2\,|\psi|^2 > 0 \qquad (1.11)
\end{aligned}$$

Because this is positive definite, the Hamiltonian $H = \int \mathrm{d}^3x\,\mathcal{H}$ is also positive definite. Therefore, in the classical field theory, there are no negative energies of the Klein-Gordon field!

As a check of (1.11), we can insert the solutions (1.4) and (1.5) into (1.11), using the normalization factor given by (1.9), and find

$$\mathcal{H}^{(+)} = \mathcal{H}^{(-)} = \frac{E_p}{V}$$

Wave equations for particles with spin 1:

The wave equations for the massless ($m = 0$) spin-1 field are the Maxwell equations, and for the massive ($m > 0$) spin-1 field the Proca equations.

(1) Maxwell equations (in vacuum)

The first set of Maxwell equations for the electric and magnetic fields is

$$\begin{aligned} \vec{\nabla} \times \vec{E} + \dot{\vec{B}} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned} \tag{1.12}$$

The second set of Maxwell equations is

$$\begin{aligned} \vec{\nabla} \times \vec{B} - \dot{\vec{E}} &= 0 \\ \vec{\nabla} \cdot \vec{E} &= 0 \end{aligned} \tag{1.13}$$

The 4-vector potential $A^\mu = \left(\phi, \vec{A}\right)$ is defined by the equations (see No. 7)

$$\vec{E} = -\nabla\phi - \dot{\vec{A}}, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{1.14}$$

Then the first set of equations (1.12) is satisfied automatically!
In order to express the equations (1.13) in terms of the vector potential, we use the field strength tensor $F^{\mu\nu}$ defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.15}$$

The components $F^{\mu\nu}$ are related to the electric and magnetic fields by (see Eq.(1.14)

$$F^{i0} = -\nabla^i A^0 - \partial^0 A^i = E^i\,, \qquad F^{ij} = \partial^i A^j - \partial^j A^i = -\left(\vec{\nabla} \times \vec{A}\right)^k = -B^k$$

[(i, j, k) is a cyclic permutation of $(1, 2, 3)$.] Then the second set of Maxwell equations (1.13) can be expressed in the compact form

$$\partial_\nu F^{\nu\mu} = 0 \tag{1.16}$$

because of $\partial_i F^{i0} = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0$ and $\partial_0 F^{0i} + \partial_j F^{ji} = 0 \Rightarrow -\dot{E}^i + \left(\nabla^j B^k - \nabla^i B^j\right) = 0$, which gives $\vec{\nabla} \times \vec{B} - \dot{\vec{E}} = 0$.