1 Massive spin-1 field equation (Proca equation)

If we add a “mass term” (similar to the Klein-Gordon equation) \( m^2 A^\mu \) to the Maxwell equation (1.16), we get the Proca equation

\[
\partial_\nu F^{\mu\nu} + m^2 A^\mu = 0
\]

\( \Rightarrow \Box A^\mu - \partial^\mu (\partial \cdot A) + m^2 A^\mu = 0 \) (1.1)

If we apply \( \partial_\mu \) to this equation, we get the relation \( \partial_\mu A^\mu = 0 \). For the massless case (Maxwell equation), this was only a choice of gauge, but for the massive case it must be satisfied. Therefore the Proca equations (1.1) are equivalent to the following set of equations:

\[
(\Box + m^2) A^\mu = 0 \quad (1.2)
\]

\[
\partial_\mu A^\mu = 0 \quad (1.3)
\]

Relations like (1.3) are called constraints. Because of the constraint, there are 3 independent components (degrees of freedom) of the field \( A^\mu \): 4 (components of \( A^\mu \)) - 1 (constraint) = 3 (degrees of freedom), as it should be for a massive spin-1 particle: The component of the spin vector along the “spin quantization axis” (we will use the \( z \)-axis) has 3 possible values \( \lambda = -1, 0, +1 \).

Plane wave solutions of the Proca equation:

The solutions with definite momentum \( \vec{p} \) and spin component \( \lambda = -1, 0, +1 \) are

\[
A^\mu(x) = \varepsilon^\mu(\vec{p}, \lambda) e^{-i(Et - \vec{p} \cdot \vec{x})} \quad (1.4)
\]

Here \( E = \pm \sqrt{\vec{p}^2 + m^2} = \pm E_p \) because of (1.2). Similar to the Klein-Gordon case, we call the solution with \( E = +E_p \) the “positive frequency solution”, and the other with \( E = -E_p \) the “negative frequency solution”. (We will show later that both solutions have positive energy.) \( \varepsilon^\mu(\vec{p}, \lambda) \) is the spin part of the wave function, called the “polarization 4-vector”, which must satisfy the constraint (from Eq.(1.3)

\[
p_\mu \varepsilon^\mu(\vec{p}, \lambda) = 0 \quad (1.5)
\]

(i) In the rest frame of the particle \( p_\mu = (m, \vec{0}) \), the polarization vector has the form

\[
\varepsilon^\mu(\vec{p} = 0, \lambda) = (0, \vec{\epsilon}_\lambda) \quad (1.6)
\]
Here the 3 vectors \( \bar{\epsilon}_{-1}, \bar{\epsilon}_0, \bar{\epsilon}_1 \) are eigenvectors of the spin operator \( \hat{S}_3 \) of a spin-1 particle with eigenvalues \(-1, 0, +1\). Here we use the following spin matrices \( \hat{S} = (\hat{S}_1, \hat{S}_2, \hat{S}_3) \) for a spin-1 particle ("adjoint representation"): 
\[
(\hat{S}_i)_{jk} = -i\epsilon_{ijk},
\]
which satisfy the commutation relations 
\[
[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k.
\]
The explicit forms are
\[
\begin{align*}
\hat{S}_1 &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0 \end{pmatrix}, \\
\hat{S}_2 &= \begin{pmatrix} 0 & 0 & i \\
0 & 0 & 0 \\
-i & 0 & 0 \end{pmatrix}, \\
\hat{S}_3 &= \begin{pmatrix} 0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}
\end{align*}
\] (1.7)

The eigenvectors of \( \hat{S}_3 \) with eigenvalues \( \lambda = -1, 0, +1 \) are then obtained as
\[
\bar{\epsilon}_{-1} = \frac{1}{\sqrt{2}} (1, -i, 0), \quad \bar{\epsilon}_0 = (0, 0, 1), \quad \bar{\epsilon}_{+1} = \frac{-1}{\sqrt{2}} (1, i, 0)
\] (1.8)

They satisfy the orthogonality and completeness relations
\[
\bar{\epsilon}_{\lambda}^\dagger \cdot \bar{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'}, \quad \sum_{\lambda} \bar{\epsilon}_{\lambda}^\dagger \bar{\epsilon}_{\lambda} = \delta_{ij}
\] (1.9)

(ii) In the frame where the particle has momentum \( \vec{p} \), we must apply a Lorentz transformation with velocity \( \vec{v} = -\vec{p}/E_p \) to the 4-vectors (1.6):
\[
\bar{\epsilon}^\mu(\vec{p}, \lambda) = \Lambda^\mu_\nu(\vec{v}) \bar{\epsilon}_\nu(\vec{p} = 0, \lambda) = \left( \frac{\vec{p} \cdot \bar{\epsilon}_{\lambda}}{m}, \bar{\epsilon}_{\lambda} + \frac{\vec{p}(\vec{p} \cdot \bar{\epsilon}_{\lambda})}{m(E_p + m)} \right)
\] (1.10)

By construction, they satisfy the following relations (see (1.5) and (1.9)):
\[
p_\mu \bar{\epsilon}^\mu(\vec{p}, s) = 0, \quad \bar{\epsilon}^\mu*(p, \lambda') \bar{\epsilon}_\mu(p, \lambda) = -\delta_{\lambda\lambda'}, \quad \bar{\epsilon}_\mu(\vec{p}, \lambda_2) = (1 - \lambda_2) \bar{\epsilon}_\mu(\vec{p}, -\lambda_2)
\] (1.11)
\[
\sum_{\lambda} \bar{\epsilon}^\mu*(p, \lambda) \bar{\epsilon}_\nu(p, \lambda) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}
\] (1.12)

\section{Lagrangian and Hamiltonian for the Proca equation}

The Proca equations (1.2), (1.3) follow from the following Lagrangian density:
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu
\]
\[
= -\frac{1}{2} (\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2} (\partial_\mu A_\nu)(\partial^\nu A^\mu) + \frac{m^2}{2} A^2
\] (2.13)
Check this:

\[
\frac{\partial L}{\partial (\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^\mu\nu
\]
\[
\frac{\partial L}{\partial A_\nu} = m^2 A^\nu
\]

and therefore the Euler-Lagrange equation for \( A^\nu \)

\[
\partial_\mu \frac{\partial L}{\partial (\partial_\mu A_\nu)} = \frac{\partial L}{\partial A_\nu}
\]

becomes the Proca equation (1.1).

Using the definition of the field strength tensor (see No. 10), the Lagrangian density (2.13) can be expressed as

\[
\mathcal{L} = \frac{1}{2} \left( \vec{E}^2 - \vec{B}^2 \right) + \frac{m^2}{2} \left( A_0^2 - \vec{A}^2 \right)
\]  

(2.14)

where (see No. 10)

\[
\vec{E} = -\vec{\nabla} A_0 - \dot{\vec{A}}, \quad \vec{B} = \vec{\nabla} \times \vec{A}
\]

The canonical momenta conjugate to \( A_0 \) and \( \vec{A} \) are then obtained as

\[
\Pi^0 \equiv \frac{\partial L}{\partial \dot{A}_0} = 0, \quad \vec{\Pi} \equiv \frac{\partial L}{\partial \dot{\vec{A}}} = -\vec{E}
\]  

(2.15)

Then the Hamiltonian density becomes

\[
\mathcal{H} = \Pi^0 \dot{A}^0 + \vec{\Pi} \cdot \dot{\vec{A}} - \mathcal{L} = -\vec{E} \cdot \dot{\vec{A}} - \mathcal{L}
\]

\[
= \left( \vec{E} + \vec{\nabla} A_0 \right) \cdot \vec{E} - \frac{1}{2} \left( \vec{E}^2 - \vec{B}^2 \right) - \frac{m^2}{2} \left( A_0^2 - \vec{A}^2 \right)
\]

\[
= \frac{1}{2} \left( \vec{E}^2 + \vec{B}^2 + m^2 \vec{A}^2 \right) + \vec{E} \cdot \vec{\nabla} A_0 - \frac{m^2}{2} A_0^2
\]  

(2.16)

The field \( A^0 \) can be eliminated by using the Proca field equation (1.1) for \( \nu = 0 \):

\[
\partial_i F^{i0} + m^2 A^0 = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = -m^2 A^0
\]  

(2.17)

Then the last term in (2.16) can be written in the form

\[
\vec{E} \cdot \vec{\nabla} A_0 = \vec{\nabla} \cdot \left( \vec{E} A_0 \right) - A_0 \left( \vec{\nabla} \cdot \vec{E} \right) = \vec{\nabla} \cdot \left( \vec{E} A_0 \right) + m^2 A_0^2
\]
Finally, the Hamiltonian becomes\footnote{The total derivative $\nabla \cdot (E A_0)$ gives a surface term which vanishes after integration.}
\[
\mathcal{H} = \int d^3 x \mathcal{H} = \frac{1}{2} \int d^3 x \left[ \vec{E}^2 + \vec{B}^2 + m^2 \left( A_0^2 + \vec{A}^2 \right) \right] \tag{2.18}
\]
This is positive definite, and therefore there are no negative energies for the Proca field. The independent (dynamical) fields are $\vec{A}$ and $\vec{E}$, while $A_0$ and $\vec{B}$ should be expressed as
\[
A_0 = -\frac{\nabla \cdot \vec{E}}{m^2}, \quad \vec{B} = \nabla \times \vec{A}
\]
In quantum field theory, the fields $\vec{A}$ and $\vec{E}$ become the dynamical quantum fields.

3 Spin 3/2 field (Rarita-Schwinger field)

We first discuss the solutions in momentum space (for definite momentum $\vec{p}$ and spin projection $s = -3/2, -1/2, +1/2, +3/2$), and then the wave equation which is satisfied by them.

If we use the Clebsch-Gordan coefficients to couple the (positive energy) spin-1/2 Dirac spinor $u(\vec{p}, s_1)$ and the spin-1 polarization vector $\epsilon^\mu(\vec{p}, s_2)$ to give spin 3/2, we obtain
\[
u^\mu(\vec{p}, s) = \sum_{s_1, s_2} \left( \frac{1}{2} 1, s_1 s_2 \right) \frac{3}{2} s \ u(\vec{p}, s_1) \ \epsilon^\mu(\vec{p}, s_2) \tag{3.19}
\]
Because this quantity has 4 Dirac spinor components and 4 vector (Lorentz) components, we can call it a “vector-spinor”. By construction it satisfies the relation $p^\mu u^\mu(\vec{p}, s) = 0$ for fixed Dirac index (see (1.5), and the Dirac equation $(\not{p} - m) u^\mu(\vec{p}, s) = 0$ for fixed Lorentz index.

In the rest system, the vector-spinors (3.19) take the form (see Eq.(1.6)
\[
u^\mu(\vec{p} = 0, s) = (0, \bar{u}(\vec{p} = 0, s)) \tag{3.20}
\]
If the particle moves with momentum $\vec{p}$, one can apply a Lorentz transformation to (3.20) with velocity $\vec{v} = -\vec{p}/E_p$:
\[
u_\sigma^\mu(\vec{p}, s) = \Lambda_\sigma^\mu(\vec{v}) \hat{S}_{ab} u_\kappa^\nu(\vec{p} = 0, s) \tag{3.21}
\]
where $\Lambda^\mu_\nu$ is the usual Lorentz matrix which acts on the polarization 4-vector, and $\hat{S}_{ab}$ is the spinor Lorentz transformation which acts on the Dirac spinor $u$. (Here $a, b = 1, 2, 3, 4$ are Dirac indices.)

Using the values for the Clebsch-Gordan coefficients, we can show the following relation:
\[
\vec{\gamma} \cdot \bar{u}(\vec{p} = 0, s) = 0 \tag{3.22}
\]
In order to show this, we first write down the explicit form of the vector-spinors (3.20): In the rest system, the polarization vectors have the form (1.6), and the Dirac spinor has the form $u(\vec{p} = 0, s) = \begin{pmatrix} \chi_s \end{pmatrix}$, where $\chi_{s=+1/2} \equiv \chi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{s=-1/2} \equiv \chi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We then obtain

$$u(\vec{p} = 0, s = 3/2) = \bar{\epsilon}_{+1} \begin{pmatrix} \chi_{\uparrow} \\ 0 \end{pmatrix}$$

$$u(\vec{p} = 0, s = 1/2) = \frac{1}{\sqrt{3}} \bar{\epsilon}_{+1} \begin{pmatrix} \chi_{\downarrow} \\ 0 \end{pmatrix} + \frac{2}{3} \bar{\epsilon}_0 \begin{pmatrix} \chi_{\uparrow} \\ 0 \end{pmatrix}$$

$$u(\vec{p} = 0, s = -1/2) = \frac{1}{\sqrt{3}} \bar{\epsilon}_{-1} \begin{pmatrix} \chi_{\uparrow} \\ 0 \end{pmatrix} + \frac{2}{3} \bar{\epsilon}_0 \begin{pmatrix} \chi_{\downarrow} \\ 0 \end{pmatrix}$$

$$u(\vec{p} = 0, s = -3/2) = \bar{\epsilon}_{-1} \begin{pmatrix} \chi_{\downarrow} \\ 0 \end{pmatrix}$$

(3.23)

Then, to show (3.22), we can use $\bar{\gamma} = \begin{pmatrix} 0 \\ -\sigma \end{pmatrix}$, and

$$\bar{\sigma} \cdot \epsilon_{+1} = -\frac{1}{\sqrt{2}} (\sigma_1 + i\sigma_2) = -\sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \equiv -\sqrt{2} \sigma_+,$$

$$\bar{\sigma} \cdot \epsilon_0 = \sigma_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\bar{\sigma} \cdot \epsilon_{-1} = -\frac{1}{\sqrt{2}} (\sigma_1 - i\sigma_2) = \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv \sqrt{2} \sigma_-.$$

Then we get:

$$\bar{\gamma} \cdot u(\vec{p} = 0, s = 3/2) = \begin{pmatrix} 0 \\ -\sqrt{2} \sigma_+ \chi_{\uparrow} \end{pmatrix} = 0$$

$$\bar{\gamma} \cdot u(\vec{p} = 0, s = 1/2) = \begin{pmatrix} 0 \\ \sqrt{2} \sigma_+ \chi_{\downarrow} - \sqrt{2} \sigma_3 \chi_{\uparrow} \end{pmatrix} = 0$$

$$\bar{\gamma} \cdot u(\vec{p} = 0, s = -1/2) = \begin{pmatrix} 0 \\ -\sqrt{2} \sigma_- \chi_{\downarrow} + \sqrt{2} \sigma_3 \chi_{\uparrow} \end{pmatrix} = 0$$

$$\bar{\gamma} \cdot u(\vec{p} = 0, s = -3/2) = \begin{pmatrix} 0 \\ -\sqrt{2} \sigma_- \chi_{\downarrow} \end{pmatrix} = 0$$

Therefore Eq.(3.22) is OK, i.e., in the rest system the relation

$$\gamma_\mu u^\mu(\vec{p} = 0, s) = 0$$

(3.24)

holds. Then by using the Lorentz transformation (3.21), we can show that also for non-zero momentum

$$\gamma_\mu u^\mu(\vec{p}, s) = (\gamma_\mu \Lambda^\mu_\nu(\vec{v})) \hat{S}(\vec{v}) u^\nu(\vec{p} = 0, s)$$

$$= (\hat{S}(\vec{v}) \gamma_\nu \hat{S}(\vec{v})^{-1}) \hat{S}(\vec{v}) u^\nu(\vec{p} = 0, s) = \hat{S}(\vec{v}) \gamma_\nu u^\nu(\vec{p} = 0, s) = 0$$

(3.25)
Therefore, the vector-spinor $u^\mu(\vec{p}, s)$ satisfies the constraint
\[ \gamma_\mu u^\mu(\vec{p}, s) = 0 \] (3.26)

We see that the vector spinor $u^\mu(\vec{p}, s)$, defined in Eq.(3.19), satisfies the following set of equations:
\begin{align*}
(\not{p} - m) \ u^\mu(\vec{p}, s) &= 0 \quad (3.27) \\
p_\mu u^\mu(\vec{p}, s) &= 0 \quad (3.28) \\
\gamma_\mu u^\mu(\vec{p}, s) &= 0 \quad (3.29)
\end{align*}

Count the number of independent components (degrees of freedom) of $u^\mu$:

- From the definition (3.19): 2 (from spin 1/2 spinor) \times 4 (from polarization 4-vector $\epsilon^\mu$) = 8 degrees of freedom
- 2 constraints from (3.28) (because (3.28) holds for each of the 2 independent spin-1/2 components)
- 2 constraints from (3.29) (because (3.29) holds for each of the 2 independent spin-1/2 components)

Therefore, there are $8 - 2 - 2 = 4$ independent degrees of freedom, which correspond to spin 3/2. (A particle with spin 3/2 can have 4 possible values of the spin component $s = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.)

In coordinate space: If we multiply the plane wave $e^{-i(E_pt - \vec{p}\cdot\vec{x})}$, we get the wave function in the form
\[ \psi^\mu(\vec{x}, t) = N(p) \ u^\mu(\vec{p}, s) e^{-i(E_pt - \vec{p}\cdot\vec{x})} \] (3.30)

where $N(p)$ is a normalization factor. This wave function satisfies the following set of equations:
\begin{align*}
(i\vec{\nabla} - m) \ \psi^\mu(\vec{x}, t) &= 0 \quad (3.31) \\
\partial_\mu \psi^\mu(\vec{x}, t) &= 0 \quad (3.32) \\
\gamma_\mu \psi^\mu(\vec{x}, t) &= 0 \quad (3.33)
\end{align*}

These are called the Rarita-Schwinger equations. Note that (3.31) is an equation of motion, and (3.32) and (3.33) are constraints.
Note: Like for the Proca equation (see Eq.(1.1)), it is possible to give one equation of motion, from which the constraint equations can be derived. This equation has the following form:

\[
(\varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\rho \partial_\rho + mg^\mu) \psi_\sigma(\vec{x}, t) = 0 \tag{3.34}
\]

Here $\varepsilon^{\mu\nu\rho\sigma}$ is the antisymmetric Levi-Civita symbol. If we contact (3.34) with $\partial_\mu$ and $\gamma_\mu$, we obtain the constraints (3.32) and (3.33) for a massive particle ($m > 0$).

4 Spin 2 field

A spin-2 particle with spin component $\lambda = -2, -1, 0, +1, +2$ can be described by a symmetric Lorentz tensor $\varepsilon^{\mu\nu}(\vec{p}, \lambda)$ which satisfies 5 constraints: The number of independent components (degrees of freedom) are then 10 (symmetric Lorentz tensor) - 5 = 5, which corresponds to the 5 possible spin orientations.

If we use the Clebsch-Gordan coefficients to couple two spin-1 polarization 4-vectors $\varepsilon^{\mu\nu}(\vec{p}, \lambda)$ to give spin 2, we obtain the following Lorentz tensor:

\[
\varepsilon^{\mu\nu}(\vec{p}, \lambda) = \sum_{\lambda_1, \lambda_2} (1 1, \lambda_1 \lambda_2 | 2 \lambda) \varepsilon^{\mu}(\vec{p}, \lambda_1) \varepsilon^{\nu}(\vec{p}, \lambda_2) \tag{4.35}
\]

Because of the symmetry property of the Clebsch-Gordan coefficient, this is a symmetric tensor: $\varepsilon^{\mu\nu}(\vec{p}, \lambda) = \varepsilon^{\nu\mu}(\vec{p}, \lambda)$. By construction, it satisfies $p_\mu \varepsilon^{\mu\nu}(\vec{p}, \lambda) = 0$ for fixed $\nu$, which are 4 constraints.

To find one more constraint, consider the following contraction of the Lorentz indices $\mu$ and $\nu$:

\[
\varepsilon_{\mu}(\vec{p}, \lambda) = \sum_{\lambda_1, \lambda_2} (1 1, \lambda_1 \lambda_2 | 2 \lambda) \varepsilon^{\mu}(\vec{p}, \lambda_1) \varepsilon_{\mu}(\vec{p}, \lambda_2)
\]

Using the relation $\varepsilon_{\mu}(\vec{p}, \lambda_2) = (-1)^{\lambda_2} \varepsilon^{*}_{\mu}(\vec{p}, -\lambda_2)$, and the orthogonality of the polarization 4-vectors (see Eq.(1.11)), this becomes

\[
\varepsilon_{\mu}(\vec{p}, \lambda) = - \sum_{\lambda_1} (1 1, \lambda_1 - \lambda_1 | 2 \lambda) (-1)^{\lambda_1}
\]

\[
= -\delta_{\lambda_0} \sum_{\lambda_1} (1 1, \lambda_1 - \lambda_1 | 2 0) (-1)^{\lambda_1} = -\delta_{\lambda_0} \left( -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} + \sqrt{2/3} \right) = 0
\]

Therefore the 5th constraint is found as

\[
\varepsilon_{\mu}(\vec{p}, \lambda) = 0 \tag{4.36}
\]
Multiplying the plane waves for a particle with energy $E_p$ and momentum $\vec{p}$, the resulting wave function $\psi^{\mu\nu}(\vec{x}, t)$ satisfies the following set of equations:

\[
(\square + m^2) \psi^{\mu\nu}(\vec{x}, t) = 0
\]

\[
\psi^{\mu\nu}(\vec{x}, t) = \psi^{\nu\mu}(\vec{x}, t)
\]

\[
\partial_\mu \psi^{\mu\nu}(\vec{x}, t) = 0
\]

\[
\psi^{\mu\nu}(\vec{x}, t) = 0
\]

The last 2 equations are constraints.

Notes: (1) It is possible to give one equation of motion for the symmetric tensor field $\psi^{\mu\nu}$, from which the constraints can be derived.
(2) It is possible to write down the field equations for general spin, by successive coupling of spin 1/2 Dirac spinors. These equations are called the Bargmann-Wigner equations. However, they are not very convenient for actual calculations.