1 4-vectors, Lorentz transformation

4-vector (example):

$$x^\mu = \left(x^0, x^1, x^2.x^3\right) \equiv \left(x^0, \vec{x}\right) \tag{1.1}$$

Here $x^0 = ct$, and the index $\mu = 0, 1, 2, 3$.

x^μ is a "space-time" 4-vector.

Lorentz transformation in x direction, velocity $\vec{v} = (v, 0, 0)$:

$$\begin{aligned} x'^0 &= \frac{x^0 - \frac{v}{c}x^1}{\sqrt{1 - v^2/c^2}} \\ x'^1 &= \frac{x^1 - \frac{v}{c}x^0}{\sqrt{1 - v^2/c^2}} \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

In matrix notation:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1.2}$$

Here $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Compact notation of Lorentz transformation:

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu(\vec{v})\, x^\nu \equiv \Lambda^\mu{}_\nu(\vec{v})\, x^\nu \tag{1.3}$$

Lorentz matrix $\Lambda^\mu{}_\nu(\vec{v})$ given in (1.2). It is a *symmetric* matrix.

$\mu = (0, 1, 2, 3)$ labels the rows, and $\nu = (0, 1, 2, 3)$ labels the columns.

In Eq.(1.3), μ is fixed, and summation over ν is implied ("Einstein convention").

Any quantity a^μ which transforms like (1.3) is called a 4-vector.

x^μ of Eq.(1.1) is called "contravariant 4-vector".

The "covariant 4-vector" x_μ is defined as

$$x_\mu = (x_0, x_1, x_2.x_3) \equiv \left(x^0, -\vec{x}\right) \tag{1.4}$$

Here $x_0 = x^0 = ct$, $x_i = -x^i$ $(i = 1, 2, 3)$.

Connection between x^μ and x_μ is given by

$$x^\mu = \sum_{\nu=0}^{3} g^{\mu\nu}\, x_\nu \equiv g^{\mu\nu}\, x_\nu$$

with the "metric tensor"

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.5}$$

Note: x^μ and x_μ are same for $\mu = 0$, and have different sign for $\mu = 1, 2, 3$.

Same rule holds for matrices (tensors). For example,

$$\Lambda_\mu{}^\nu(\vec{v}) = \begin{pmatrix} \gamma & \gamma\frac{v}{c} & 0 & 0 \\ \gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.6}$$

Note: The *first* index labels the rows, and the *second* index labels the columns of a matrix.

The Lorentz transformation (1.3) can be expressed also for covariant 4-vectors:

$$x'_\mu \;=\; \sum_{\nu=0}^{3} \Lambda_\mu{}^\nu(\vec{v})\, x_\nu \tag{1.7}$$

Comparing (1.2) and (1.6), we see that

$$\Lambda_\mu{}^\nu(\vec{v}) = \Lambda^\nu{}_\mu(-\vec{v})$$

<u>Important property of Lorentz matrix</u>:

$$\sum_\mu \Lambda_\mu{}^\nu(\vec{v})\, \Lambda^\mu{}_\sigma(\vec{v}) = \sum_\mu \Lambda^\nu{}_\mu(-\vec{v})\, \Lambda^\mu{}_\sigma(\vec{v}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = g^\nu{}_\sigma \tag{1.8}$$

Definition of "<u>scalar product</u>" (S) of two 4-vectors a and b:

$$S = \sum_\mu a_\mu\, b^\mu \equiv a \cdot b = a^0\, b^0 - \vec{a} \cdot \vec{b}$$

Because of (1.8), this is invariant under Lorentz transformations:

$$S' = a'_\mu \, b'^\mu = \left(\Lambda_\mu{}^\nu a_\nu\right) \, \left(\Lambda^\mu{}_\sigma b^\sigma\right) = \left(\Lambda_\mu{}^\nu \, \Lambda^\mu{}_\sigma\right) \; a_\nu \, b^\sigma = g^\nu{}_\sigma a_\nu \, b^\sigma = a_\nu \, b^\nu = S$$

Example: $x \cdot p = x^0 \, p^0 - \vec{x} \cdot \vec{p}$ is Lorentz invariant.

Other example of 4-vector: <u>Momentum 4-vector</u>

$$p^\mu = \left(p^0, \vec{p}\right) = \left(\frac{E_p}{c}, \vec{p}\right)$$

Here $E_p = \sqrt{(pc)^2 + (mc^2)^2}$ is the energy of a free particle with momentum $p = |\vec{p}|$.

<u>Lorentz transformation of the "4-derivative"</u>

$$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \qquad (1.9)$$

(1) The inverse Lorentz transformation of x is

$$x^\nu = \Lambda^\nu{}_\mu(-\vec{v}) \, x'^\mu$$

From this we obtain

$$\frac{\partial x^\nu}{\partial x'^\mu} = \Lambda^\nu{}_\mu(-\vec{v}) = \Lambda_\mu{}^\nu(\vec{v})$$

(2) Therefore, using the chain rule,

$$\frac{\partial}{\partial x'^\mu} = \sum_\nu \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \sum_\nu \Lambda_\mu{}^\nu(\vec{v}) \, \frac{\partial}{\partial x^\nu}$$

Therefore the 4-derivative $\frac{\partial}{\partial x^\nu}$ transforms like x_ν, i.e., like a *covariant* 4-vector !

We therefore define the "<u>covariant derivative</u>" as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

Then

$$\partial'_\mu = \Lambda_\mu{}^\nu(\vec{v}) \, \partial_\nu$$

The "contravariant derivative" is then

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

Its Lorentz transformation is the same as for x^μ, i.e.,

$$\partial'^{\mu} = \Lambda^{\mu}{}_{\nu}(\vec{v})\,\partial'^{\nu}$$

The 4-divergence of a 4-vector a is defined by

$$\partial_\mu a^\mu = \frac{\partial a^0}{\partial x^0} + \vec{\nabla}\cdot\vec{a}$$

This is Lorentz invariant.

The d'Alembert operator is defined by

$$\partial^2 = \partial_\mu\,\partial^\mu = \Box = \frac{\partial^2}{\partial x^{02}} - \Delta$$

Here Δ is the Laplace operator. The d'Alembert operatror is Lorentz invariant.