3 Wave equations for particles with spin 1

The wave equations for the massless ($m = 0$) spin-1 field are the Maxwell equations, and for the massive ($m > 0$) spin-1 field the Proca equations.

3.1 Maxwell equations (in vacuum)

The first set of Maxwell equations for the electric and magnetic fields is

$$\begin{aligned} \vec{\nabla} \times \vec{E} + \dot{\vec{B}} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned} \tag{3.1}$$

The second set of Maxwell equations is

$$\begin{aligned} \vec{\nabla} \times \vec{B} - \dot{\vec{E}} &= 0 \\ \vec{\nabla} \cdot \vec{E} &= 0 \end{aligned} \tag{3.2}$$

The 4-vector potential $A^\mu = \left(\phi, \vec{A}\right)$ is defined by the equations (see Sect. 10 or RQM1)

$$\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{3.3}$$

Then the first set of equations (3.1) is satisfied automatically!

In order to express the equations (3.2) in terms of the vector potential, we use the field strength tensor $F^{\mu\nu}$ defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.4}$$

The components $F^{\mu\nu}$ are related to the electric and magnetic fields by (see Eq.(3.3))

$$F^{i0} = -\nabla^i A^0 - \partial^0 A^i = E^i, \qquad F^{ij} = \partial^i A^j - \partial^j A^i = -\left(\vec{\nabla} \times \vec{A}\right)^k = -B^k$$

[(i, j, k) is a cyclic permutation of $(1, 2, 3)$.] Then the second set of Maxwell equations (3.2) can be expressed in the compact form

$$\partial_\nu F^{\nu\mu} = 0 \Leftrightarrow \Box A^\mu - \partial^\mu (\partial \cdot A) = 0 \tag{3.5}$$

because of: $\partial_i F^{i0} = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0$ and $\partial_0 F^{0i} + \partial_j F^{ji} = 0 \Rightarrow -\dot{E}^i + \left(\nabla^j B^k - \nabla^k B^j\right) = 0$, which gives $\vec{\nabla} \times \vec{B} - \dot{\vec{E}} = 0$. One can choose a gauge ("Lorentz gauge") where $\partial \cdot A = 0$, then (3.5) simplifies to $\Box A^\mu = 0$.

3.2 Massive spin-1 field equation (Proca equation)

If we add a "mass term" (similar to the Klein-Gordon equation) $m^2 A^\mu$ to the Maxwell equation (3.5), we get the Proca equation

$$\begin{aligned} \partial_\nu F^{\nu\mu} + m^2 A^\mu &= 0 \\ \Rightarrow \Box A^\mu - \partial^\mu (\partial \cdot A) + m^2 A^\mu &= 0 \end{aligned} \tag{3.6}$$

If we apply ∂_μ to this equation, we get the relation $\partial_\mu A^\mu = 0$. For the massless case (Maxwell equation), this was only a choice of gauge, but for the massive case it must be satisfied. Therefore the Proca equations (3.6) are equivalent to the following set of equations:

$$\left(\Box + m^2\right) A^\mu = 0 \tag{3.7}$$

$$\partial_\mu A^\mu = 0 \tag{3.8}$$

Relations like (3.8) are called constraints. Because of the constraint, there are three independent components (degrees of freedom) of the field A^μ, namely 4 (components of A^μ) - 1 (constraint) = 3 (degrees of freedom), as it should be for a massive spin-1 particle: The component of the spin vector along the "spin quantization axis" (we will use the z-axis) has three possible values $\lambda = -1, 0, +1$.

Plane wave solutions of the Proca equation:

The solutions with definite momentum $\vec{p}$ and spin component $\lambda = -1, 0, +1$ are

$$A^\mu(x) = \varepsilon^\mu(\vec{p}, \lambda)\, e^{-i(Et - \vec{p}\cdot\vec{x})} \tag{3.9}$$

Here $E = \pm\sqrt{\vec{p}^2 + m^2} = \pm E_p$ because of (3.7). Similar to the Klein-Gordon case, we call the solution with $E = +E_p$ the "positive frequency solution", and the other with $E = -E_p$ the "negative frequency solution". (We will show later that both solutions have positive energy.) $\varepsilon^\mu(\vec{p}, \lambda)$ is the spin part of the wave function, called the "polarization 4-vector", which must satisfy the constraint (from Eq.(3.8))

$$p_\mu\, \varepsilon^\mu(\vec{p}, \lambda) = 0 \tag{3.10}$$

(i) In the rest frame of the particle $p_\mu = (m, \vec{0})$, the polarization vector has the form

$$\varepsilon^\mu(\vec{p} = 0, \lambda) = (0, \vec{\epsilon}_\lambda) \tag{3.11}$$

Here the set of three vectors $(\vec{\epsilon}_{-1}, \vec{\epsilon}_0, \vec{\epsilon}_1)$ is not determined by the Proca equation, but can be chosen as eigenvectors of the z-component of the spin operator ($\hat{S}_3$) with eigenvalues $-1, 0, +1$. Here we use the following spin matrices $\hat{\vec{S}} = \left(\hat{S}_1, \hat{S}_2, \hat{S}_3\right)$ for a spin-1 particle ("adjoint representation"): $\left(\hat{S}_i\right)_{jk} = -i\epsilon_{ijk}$, which satisfy the commutation relations $\left[\hat{S}_i, \hat{S}_j\right] = i\epsilon_{ijk}\hat{S}_k$. The explicit forms are

$$\hat{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \hat{S}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.12}$$

The eigenvectors of $\hat{S}_3$ with eigenvalues $\lambda = -1, 0, +1$ are then obtained as

$$\vec{\epsilon}_{-1} = \frac{1}{\sqrt{2}}(1, -i, 0) , \quad \vec{\epsilon}_0 = (0, 0, 1) , \quad \vec{\epsilon}_{+1} = \frac{-1}{\sqrt{2}}(1, i, 0) \tag{3.13}$$

They satisfy the orthogonality and completeness relations

$$\vec{\epsilon}^{\,\dagger}_{\lambda'} \cdot \vec{\epsilon}_\lambda = \delta_{\lambda'\lambda} , \qquad \sum_\lambda \epsilon^i_\lambda \, \epsilon^{j\dagger}_\lambda = \delta_{ij} \tag{3.14}$$

(ii) In the frame where the particle has momentum $\vec{p}$, we must apply a Lorentz transformation with velocity $\vec{v} = -\vec{p}/E_p$ to the 4-vectors (3.11):

$$\varepsilon^\mu(\vec{p}, \lambda) = \Lambda^\mu{}_\nu(\vec{v})\, \varepsilon_\nu(\vec{p} = 0, \lambda) = \left(\frac{\vec{p} \cdot \vec{\epsilon_\lambda}}{m}, \, \vec{\epsilon}_\lambda + \frac{\vec{p}\,(\vec{p} \cdot \vec{\epsilon}_\lambda)}{m\,(E_p + m)}\right) \tag{3.15}$$

By construction, they satisfy the following relations (see (3.10) and (3.14)):

$$p_\mu \varepsilon^\mu(\vec{p}, s) = 0 , \qquad \varepsilon^{\mu *}(p, \lambda')\, \varepsilon_\mu(p, \lambda) = -\delta_{\lambda\lambda'} , \qquad \varepsilon_\mu(\vec{p}, \lambda) = (-1)^\lambda \, \varepsilon^*_\mu(\vec{p}, -\lambda) \tag{3.16}$$

$$\sum_\lambda \varepsilon^{\mu *}(p, \lambda)\, \varepsilon^\nu(p, \lambda) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \tag{3.17}$$

3.3 Lagrangian and Hamiltonian for the Proca equation

The Proca equations (3.7), (3.8) follow from the following Lagrangian density:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \\ &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) + \frac{m^2}{2} A^2 \end{aligned} \tag{3.18}$$

Check this:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} &= -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu} \\ \frac{\partial \mathcal{L}}{\partial A_\nu} &= m^2\, A^\nu \end{aligned}$$

and therefore the Euler-Lagrange equation for A^ν

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = \frac{\partial \mathcal{L}}{\partial A_\nu}$$

becomes the Proca equation (3.6).

Using the definition of the field strength tensor (see Eq.(3.4)), the Lagrangian density (3.18) can be expressed as

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{m^2}{2}\left(A_0^2 - \vec{A}^2\right) \tag{3.19}$$

where (see Eq.(3.3))

$$\vec{E} = -\vec{\nabla} A_0 - \dot{\vec{A}}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

The momenta conjugate to A_0 and $\vec{A}$ are then obtained as

$$\Pi^0 \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0\,, \qquad \vec{\Pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = -\vec{E} \tag{3.20}$$

Then the Hamiltonian density becomes

$$\begin{aligned} \mathcal{H} &= \Pi^0\, \dot{A}^0 + \vec{\Pi} \cdot \dot{\vec{A}} - \mathcal{L} = -\vec{E} \cdot \dot{\vec{A}} - \mathcal{L} \\ &= \left(\vec{E} + \vec{\nabla} A_0\right) \cdot \vec{E} - \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) - \frac{m^2}{2}\left(A_0^2 - \vec{A}^2\right) \\ &= \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2 + m^2 \vec{A}^2\right) - \frac{m^2}{2} A_0^2 + \vec{E} \cdot \vec{\nabla} A_0 \end{aligned} \tag{3.21}$$

The field A^0 can be eliminated by using the Proca field equation (first equation in 3.6) for $\mu = 0$:

$$\partial_i F^{i0} + m^2\, A^0 = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = -m^2\, A^0 \tag{3.22}$$

Then the last term in (3.21) can be written in the form

$$\vec{E} \cdot \vec{\nabla} A_0 = \vec{\nabla} \cdot \left(\vec{E}\, A_0\right) - A_0 \left(\vec{\nabla} \cdot \vec{E}\right) = \vec{\nabla} \cdot \left(\vec{E}\, A_0\right) + m^2\, A_0^2$$

Finally, the Hamiltonian (3.21) becomes [1]

$$\mathcal{H} = \int \mathrm{d}^3x\, \mathcal{H} = \frac{1}{2} \int \mathrm{d}^3x\, \left[\vec{E}^2 + \vec{B}^2 + m^2 \left(A_0^2 + \vec{A}^2 \right) \right] \tag{3.23}$$

This is positive definite, and therefore there are no negative energies for the Proca field. The independent (dynamical) fields are $\vec{A}$ and $\vec{E}$, while A_0 and $\vec{B}$ should be expressed as

$$A_0 = -\frac{\vec{\nabla} \cdot \vec{E}}{m^2}\,, \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

In quantum field theory, the fields $\vec{A}$ and $\vec{E}$ become the dynamical quantum fields.

[1] The total derivative $\vec{\nabla} \cdot \left(\vec{E}\, A_0 \right)$ gives a surface term which vanishes after integration.