4 Spin 3/2 field (Rarita-Schwinger field)

We first try to get the solutions in momentum space (for definite momentum $\vec{p}$ and spin projection $s = -3/2, -1/2, +1/2, +3/2$), and then discuss the wave equation which is satisfied by them.

If we use the Clebsch-Gordan coefficients[1] to couple the (positive energy) spin-1/2 Dirac spinor $u(\vec{p}, s_1)$ and the spin-1 polarization vector $\epsilon^\mu(\vec{p}, s_2)$ to give spin 3/2, we obtain

$$u^\mu(\vec{p}, s) = \sum_{s_1, s_2} \left(\frac{1}{2}\, 1,\ s_1\, s_2 | \frac{3}{2}\, s \right) u(\vec{p}, s_1)\, \epsilon^\mu(\vec{p}, s_2) \tag{4.1}$$

Because this quantity has 4 Dirac spinor components and 4 vector (Lorentz) components, we can call it a "vector-spinor". By construction it satisfies the relation $p_\mu u^\mu(\vec{p}, s) = 0$ for fixed Dirac index (see Eq.(3.10) of Sect. 3), and the Dirac equation $(\not{p} - m)\, u^\mu(\vec{p}, s) = 0$ for fixed Lorentz index.
In the rest system, the vector-spinors (4.1) take the form (see Eq.(3.11) of Sect. 3)

$$u^\mu(\vec{p} = 0, s) = (0,\ \vec{u}(\vec{p} = 0, s)) \tag{4.2}$$

If the particle moves with momentum $\vec{p}$, one can apply a Lorentz transformation to (4.2) with velocity $\vec{v} = -\vec{p}/E_p$:

$$u_a^\mu(\vec{p}, s) = \Lambda^\mu{}_\nu(\vec{v})\, \hat{S}_{ab}\, u_b^\nu(\vec{p} = 0, s) \tag{4.3}$$

where $\Lambda^\mu{}_\nu$ is the usual Lorentz matrix which acts on the polarization 4-vector, and $\hat{S}_{ab}$ is the spinor Lorentz transformation [2] which acts on the Dirac spinor u. (Here $a, b = 1, 2, 3, 4$ are Dirac indices.)

Using the values for the Glebsch-Gordon coefficients, we can show the following relation:

$$\vec{\gamma} \cdot \vec{u}(\vec{p} = 0, s) = 0 \tag{4.4}$$

In order to show this, we first write down the explicit form of the vector-spinors (4.2): In the rest system, the polarization vectors have the form of Eq.(3.13) of Sect. 3, and the Dirac spinor has the form $u(\vec{p} = 0, s) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}$, where $\chi_{s=+1/2} \equiv \chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{s=-1/2} \equiv \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We then

[1] For the definition of the Clebsch-Gordan (angular momentum coupling) coefficients, see any textbook on quantum mechanics.

[2] See Sect.8 of RQM1 for the form of the spinor Lorentz transformation.

obtain from Eq.(4.1)

$$
\begin{aligned}
\vec{u}(\vec{p}=0, s=\frac{3}{2}) &= \vec{\epsilon}_{+1}\begin{pmatrix} \chi_\uparrow \\ 0 \end{pmatrix} \\
\vec{u}(\vec{p}=0, s=\frac{1}{2}) &= \frac{1}{\sqrt{3}}\,\vec{\epsilon}_{+1}\begin{pmatrix} \chi_\downarrow \\ 0 \end{pmatrix} + \sqrt{\frac{2}{3}}\,\vec{\epsilon}_0\begin{pmatrix} \chi_\uparrow \\ 0 \end{pmatrix} \\
\vec{u}(\vec{p}=0, s=-\frac{1}{2}) &= \frac{1}{\sqrt{3}}\,\vec{\epsilon}_{-1}\begin{pmatrix} \chi_\uparrow \\ 0 \end{pmatrix} + \sqrt{\frac{2}{3}}\,\vec{\epsilon}_0\begin{pmatrix} \chi_\downarrow \\ 0 \end{pmatrix} \\
\vec{u}(\vec{p}=0, s=-\frac{3}{2}) &= \vec{\epsilon}_{-1}\begin{pmatrix} \chi_\downarrow \\ 0 \end{pmatrix} \qquad (4.5)
\end{aligned}
$$

Then, to show (4.4), we can use $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$, and

$$
\begin{aligned}
\vec{\sigma}\cdot\vec{\epsilon}_{+1} &= -\frac{1}{\sqrt{2}}\,(\sigma_1 + i\sigma_2) = -\sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv -\sqrt{2}\sigma_+\,, \\
\vec{\sigma}\cdot\vec{\epsilon}_0 &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \\
\vec{\sigma}\cdot\vec{\epsilon}_{-1} &= \frac{1}{\sqrt{2}}\,(\sigma_1 - i\sigma_2) = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv \sqrt{2}\sigma_-
\end{aligned}
$$

Then we get:

$$
\begin{aligned}
\vec{\gamma}\cdot\vec{u}(\vec{p}=0, s=\frac{3}{2}) &= \begin{pmatrix} -\sqrt{2}\sigma_+\chi_\uparrow \\ 0 \end{pmatrix} = 0 \\
\vec{\gamma}\cdot\vec{u}(\vec{p}=0, s=\frac{1}{2}) &= \begin{pmatrix} -\frac{1}{\sqrt{6}}\sigma_+\chi_\downarrow + \sqrt{\frac{2}{3}}\sigma_3\chi_\uparrow \\ 0 \end{pmatrix} = 0 \\
\vec{\gamma}\cdot\vec{u}(\vec{p}=0, s=-\frac{1}{2}) &= \begin{pmatrix} \frac{1}{\sqrt{6}}\sigma_-\chi_\uparrow + \sqrt{\frac{2}{3}}\sigma_3\chi_\downarrow \\ 0 \end{pmatrix} = 0 \\
\vec{\gamma}\cdot\vec{u}(\vec{p}=0, s=-\frac{3}{2}) &= \begin{pmatrix} \sqrt{2}\sigma_-\chi_\downarrow \\ 0 \end{pmatrix} = 0
\end{aligned}
$$

Therefore Eq.(4.4) is OK, i.e., in the rest system the relation

$$\gamma_\mu u^\mu(\vec{p}=0, s) = 0 \qquad (4.6)$$

holds. Then by using the Lorentz transformation (4.3), we can show that also for non-zero momentum

$$
\begin{aligned}
\gamma_\mu u^\mu(\vec{p}, s) &= \left(\gamma_\mu \Lambda^\mu{}_\nu(\vec{v})\right)\,\hat{S}(\vec{v})\,u^\nu(\vec{p}=0, s) \\
&= \left(\hat{S}(\vec{v})\gamma_\nu\hat{S}(\vec{v})^{-1}\right)\,\hat{S}(\vec{v})\,u^\nu(\vec{p}=0, s) = \hat{S}(\vec{v})\,\gamma_\nu u^\nu(\vec{p}=0, s) = 0 \qquad (4.7)
\end{aligned}
$$

(In the second equality, we used Eq.(6.7) of RMQ1.) Therefore, the vector-spinor $u^\mu(\vec{p},s)$ satisfies the constraint

$$\gamma_\mu u^\mu(\vec{p},s) = 0 \tag{4.8}$$

We see that the vector spinor $u^\mu(\vec{p},s)$, defined in Eq.(4.1), satisfies the following set of equations:

$$\begin{aligned}(\not{p} - m)\, u^\mu(\vec{p},s) &= 0 &(4.9)\\ p_\mu\, u^\mu(\vec{p},s) &= 0 &(4.10)\\ \gamma_\mu\, u^\mu(\vec{p},s) &= 0 &(4.11)\end{aligned}$$

Count the number of independent components (degrees of freedom) of u^μ:

- From the definition (4.1): 2 (from spin 1/2 spinor) $\times$ 4 (from polarization 4-vector ϵ^μ) $= 8$ degrees of freedom
- 2 constraints from (4.10)
  (because (4.10) holds for each of the 2 independent spin-1/2 components)
- 2 constraints from (4.11)
  (because (4.11) holds for each of the 2 independent spin-1/2 components)

Therefore, there are $8 - 2 - 2 = 4$ independent degrees of freedom, which correspond to spin 3/2. (A particle with spin 3/2 can have 4 possible values of the spin component $s = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.)

In coordinate space: If we multiply the plane wave $e^{-i(E_p t - \vec{p}\cdot\vec{x})}$, we get the wave function in the form

$$\psi^\mu(\vec{x},t) = N(p)\, u^\mu(\vec{p},s)\, e^{-i(E_p t - \vec{p}\cdot\vec{x})} \tag{4.12}$$

where $N(p)$ is a normalization factor. This wave function satisfies the following set of equations:

$$\begin{aligned}(i\not{\nabla} - m)\, \psi^\mu(\vec{x},t) &= 0 &(4.13)\\ \partial_\mu \psi^\mu(\vec{x},t) &= 0 &(4.14)\\ \gamma_\mu\, \psi^\mu(\vec{x},t) &= 0 &(4.15)\end{aligned}$$

These are called the <u>Rarita-Schwinger equations</u>. Note that (4.13) is an equation of motion, and (4.14) and (4.15) are constraints.

Note: Like for the Proca equation, it is possible to give one equation of motion, from which the constraint equations can be derived. (For the Proca equation, this was given by Eq.(3.6) of Sect. 3.) This equation has the following form:

$$\left(\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho + m g^{\mu\sigma}\right)\, \psi_\sigma(\vec{x},t) = 0 \tag{4.16}$$

Here $\varepsilon^{\mu\nu\rho\sigma}$ is the antisymmetric Levi-Civita symbol. If we contact (4.16) with ∂_μ and γ_μ, we obtain the constraints (4.14) and (4.15) for a massive particle ($m > 0$).

5 Spin 2 field

A spin-2 particle with spin component $\lambda = -2, -1, 0, +1, +2$ can be described by a symmetric Lorentz tensor $\varepsilon^{\mu\nu}(\vec{p},\lambda)$ which satisfies 5 constraints: The number of independent components (degrees of freedom) are then 10 (symmetric Lorentz tensor) - 5 (constraints) = 5, which corresponds to the 5 possible spin orientations.

If we use the Clebsch-Gordan coefficients to couple two spin-1 polarization 4-vectors $\varepsilon^{\mu\nu}(\vec{p},\lambda)$ to give spin 2, we obtain the following Lorentz tensor:

$$\varepsilon^{\mu\nu}(\vec{p},\lambda) = \sum_{\lambda_1,\lambda_2} (1\,1,\ \lambda_1\,\lambda_2|2\,\lambda)\ \varepsilon^{\mu}(\vec{p},\lambda_1)\,\varepsilon^{\nu}(\vec{p},\lambda_2) \tag{5.1}$$

Because of the symmetry property of the Clebsch-Gordan coefficient, this is a symmetric tensor: $\varepsilon^{\mu\nu}(\vec{p},\lambda) = \varepsilon^{\nu\mu}(\vec{p},\lambda)$. By construction, it satisfies $p_\mu\,\varepsilon^{\mu\nu}(\vec{p},\lambda) = 0$ for fixed ν, which are 4 constraints. To find one more constraint, consider the following contraction of the Lorentz indices μ and ν:

$$\varepsilon^{\mu}{}_{\mu}(\vec{p},\lambda) = \sum_{\lambda_1,\lambda_2} (1\,1,\ \lambda_1\,\lambda_2|2\,\lambda)\ \varepsilon^{\mu}(\vec{p},\lambda_1)\,\varepsilon_{\mu}(\vec{p},\lambda_2)$$

Using the relations (see Sect. 3) $\varepsilon_\mu(\vec{p},\lambda_2) = (-1)^{\lambda_2}\,\varepsilon^*_\mu(\vec{p},-\lambda_2)$ and $\varepsilon^*_\mu(\vec{p},\lambda')\varepsilon^\mu(\vec{p},\lambda) = -\delta_{\lambda'\lambda}$, this becomes

$$\begin{aligned}
\varepsilon^{\mu}{}_{\mu}(\vec{p},\lambda) &= -\sum_{\lambda_1} (1\,1\,,\ \lambda_1\ -\lambda_1|2\,\lambda)\ (-1)^{\lambda_1} \\
&= -\delta_{\lambda 0}\sum_{\lambda_1} (1\,1,\ \lambda_1\ -\lambda_1|2\,0)\ (-1)^{\lambda_1} = -\delta_{\lambda 0}\left(-\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} + \sqrt{\frac{2}{3}}\right) = 0
\end{aligned}$$

Therefore the 5th constraint is found as

$$\varepsilon^{\mu}{}_{\mu}(\vec{p},\lambda) = 0 \tag{5.2}$$

Multiplying the plane waves for a particle with energy E_p and momentum $\vec{p}$, the resulting wave function $\psi^{\mu\nu}(\vec{x},t)$ satisfies the following set of equations:

$$\begin{aligned}
\left(\Box + m^2\right)\, \psi^{\mu\nu}(\vec{x},t) &= 0 \\
\psi^{\mu\nu}(\vec{x},t) &= \psi^{\nu\mu}(\vec{x},t) \\
\partial_\mu \psi^{\mu\nu}(\vec{x},t) &= 0 \\
\psi^{\mu}{}_{\mu}(\vec{x},t) &= 0
\end{aligned}$$

The last 2 equations are constraints.

Notes: (1) It is possible to give one equation of motion for the symmetric tensor field $\psi^{\mu\nu}$, from which the constraints can be derived.

(2) It is possible to write down the field equations for general spin, by successive coupling of spin 1/2 Dirac spinors. These equations are called the Bargmann-Wigner equations. However, they are not very convenient for actual calculations.