4 Solutions of the free Dirac equation

Dirac equation for free particle:

$$\left[(\vec{\alpha}\cdot\hat{\vec{p}})\,c+\beta\,(mc^2)\right]\psi(\vec{x},t)=i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} \tag{4.1}$$

where $\hat{\vec{p}}=-i\hbar\vec{\nabla}$ is the momentum operator. Plane wave solution for free particle with momentum $\vec{p}$:

$$\psi(\vec{x},t)=w(\vec{p},s)\,e^{-i(Et-\vec{p}\cdot\vec{x})/\hbar} \tag{4.2}$$

where $E=\pm\sqrt{(pc)^2+(mc^2)^2}$, and $w(\vec{p},s)$ is a 4-component "Dirac spinor", which depends on the spin direction s (see later). Inserting (4.2) into (4.1),

$$\left[(\vec{\alpha}\cdot\vec{p})\,c+\beta\,(mc^2)\right]w(\vec{p},s)=Ew(\vec{p},s) \tag{4.3}$$

We express $w(\vec{p},s)$ in the form

$$w(\vec{p},s)=\begin{pmatrix}\phi\\ \chi\end{pmatrix} \tag{4.4}$$

where ϕ and χ are 2-component "Pauli spinors", depending on $(\vec{p},s)$. Inserting this into (4.3),

$$\begin{pmatrix}(E-mc^2) & -(\vec{\sigma}\cdot\vec{p})c\\ -(\vec{\sigma}\cdot\vec{p})c & (E+mc^2)\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix}=0 \tag{4.5}$$

Therefore, the coupled equations for ϕ and χ are

$$\left(E-mc^2\right)\phi-(\vec{\sigma}\cdot\vec{p})c\;\chi \;=\; 0 \tag{4.6}$$

$$\left(E+mc^2\right)\chi-(\vec{\sigma}\cdot\vec{p})c\;\phi \;=\; 0 \tag{4.7}$$

- For $E=+\sqrt{(pc)^2+(mc^2)^2}\equiv E_p$ (positive energy), we use (4.7) to eliminate χ:

  $$\chi=\frac{(\vec{\sigma}\cdot\vec{p})c}{E_p+mc^2}\,\phi \tag{4.8}$$

  Then the positive energy spinor (w_+) becomes

  $$w_+(\vec{p},s)\equiv u(\vec{p},s)=N_p\begin{pmatrix}\phi(s)\\ \frac{(\vec{\sigma}\cdot\vec{p})c}{E_p+mc^2}\,\phi(s)\end{pmatrix} \tag{4.9}$$

  Here N_p is a normalization factor (see later).

- For $E = -\sqrt{(pc)^2 + (mc^2)^2} \equiv -E_p$ (negative energy), we use (4.6) to eliminate ϕ:

$$\phi = -\frac{(\vec{\sigma}\cdot\vec{p})c}{E_p + mc^2}\,\chi \tag{4.10}$$

  Then the negative energy spinor (w_-) becomes

$$w_-(\vec{p},s) \equiv v(-\vec{p},s) = N_p \begin{pmatrix} -\frac{(\vec{\sigma}\cdot\vec{p})c}{E_p+mc^2}\,\chi(s) \\ \chi(s) \end{pmatrix} \tag{4.11}$$

The Dirac equation does not determine the 2-component Pauli spinors ϕ in (4.9) or χ in (4.11) [1] !
Possible choice of Pauli spinors: If the particle has a definite spin direction (up or down) in its rest frame: Define the z axis as the "spin quantization axis", and require that $u(\vec{p}=0,s)$ and $v(\vec{p}=0,s)$ are eigenvectors of the spin operator (in units of $\hbar$) $S_3 = \frac{1}{2}\Sigma_3$, with eigenvalues $s = \pm 1/2$:

$$\begin{aligned} S_3\, u(\vec{p}=0,s) &= s\, u(\vec{p}=0,s) \Rightarrow \frac{1}{2}\sigma_3\, \phi(s) = s\, \phi(s) \Rightarrow \phi(+1/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \phi(-1/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ S_3\, v(\vec{p}=0,s) &= s\, v(\vec{p}=0,s) \Rightarrow \frac{1}{2}\sigma_3\, \chi(s) = s\, \chi(s) \Rightarrow \chi(+1/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \chi(-1/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{4.12}$$

Notes:

- If some other direction $\vec{n}$ is chosen as the spin quantization axis, then one chooses $\phi(s)$ and $\chi(s)$ as eigenvectors of $\frac{1}{2}(\vec{\sigma}\cdot\vec{n})$ with eigenvalues $s = \pm 1/2$: $\frac{1}{2}(\vec{\sigma}\cdot\vec{n})\,\phi(s) = s\,\phi(s)$, and $\frac{1}{2}(\vec{\sigma}\cdot\vec{n})\,\chi(s) = s\,\chi(s)$.

- Important point: Hamiltonian H and spin operators $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$ do not commute: $[H, \Sigma_i] \neq 0$ ($i = 1,2,3$). $\Rightarrow$ In general, the spinors u, v are eigenvectors of H, but cannot be also eigenvectors of S_3.

- Normalization of spinors: We choose the orthonormalization as

$$u^\dagger(\vec{p},s')u(\vec{p},s) = \frac{E_p}{mc^2}\delta_{ss'}\,, \qquad v^\dagger(\vec{p},s')v(\vec{p},s) = \frac{E_p}{mc^2}\delta_{ss'}\,, \qquad v^\dagger(-\vec{p},s')u(\vec{p},s) = u^\dagger(\vec{p},s')v(-\vec{p},s) = 0 \tag{4.13}$$

[1]Reason: (1) Dirac eq. is a homogeneous matrix equation; (2) the spin direction is still no specified in (4.9) and (4.11).

This determines the normalization factor N_p. Using $\phi^\dagger(s')\phi(s) = \delta_{ss'}$ we get

$$\begin{aligned} N_p^2\,\phi^\dagger(s')\left(1+\frac{\vec{p}^2c^2}{(E_p+mc^2)^2}\right)\phi(s) &\equiv \frac{E_p}{mc^2}\delta_{ss'} \\ \Rightarrow N_p^2\left(1+\frac{\vec{p}^2c^2}{(E_p+mc^2)^2}\right) &\equiv \frac{E_p}{mc^2} \\ \Rightarrow N_p^2\left(1+\frac{E_p-mc^2}{E_p+mc^2}\right) &\equiv \frac{E_p}{mc^2} \end{aligned}$$

This gives

$$N_p = \sqrt{\frac{E_p+mc^2}{2mc^2}} \tag{4.14}$$

Finally, the wave functions (4.2) are normalized (in a volume V) as

$$\int_V \mathrm{d}^3x \;\; \psi^\dagger(\vec{x},t)\psi(\vec{x},t) = 1 \tag{4.15}$$

Final results for solutions of the Dirac equation (4.1):

- Positive energy solution:

$$\begin{aligned} \psi^{(+)}_{\vec{p},s}(\vec{x},t) &= \frac{1}{\sqrt{V}}\sqrt{\frac{mc^2}{E_p}}u(\vec{p},s)e^{-i(E_pt-\vec{p}\cdot\vec{x})/\hbar} \\ u(\vec{p},s) &= \sqrt{\frac{E_p+mc^2}{2mc^2}}\begin{pmatrix} \phi(s) \\ \frac{(\vec{\sigma}\cdot\vec{p})c}{E_p+mc^2}\phi(s) \end{pmatrix} \end{aligned} \tag{4.16}$$

  This is the wave function of a particle with energy $E_p > 0$, momentum $\vec{p}$, and spin projection $s = \pm 1/2$ in its rest frame.

- Negative energy solution:

$$\begin{aligned} \psi^{(-)}_{\vec{p},s}(\vec{x},t) &= \frac{1}{\sqrt{V}}\sqrt{\frac{mc^2}{E_p}}v(-\vec{p},s)e^{i(E_pt+\vec{p}\cdot\vec{x})/\hbar} \\ v(\vec{p},s) &= \sqrt{\frac{E_p+mc^2}{2mc^2}}\begin{pmatrix} \frac{(\vec{\sigma}\cdot\vec{p})c}{E_p+mc^2}\phi(s) \\ \phi(s) \end{pmatrix} \end{aligned} \tag{4.17}$$

  This is the wave function of a particle with energy $-E_p < 0$, momentum $\vec{p}$, and spin projection $s = \pm 1/2$ in its rest frame.

These wave functions satisfy the orthonormalization conditions (with $\alpha, \alpha' = +$ or $-$)

$$\int_V \mathrm{d}^3x\, \psi^{(\alpha')\dagger}_{\vec{p},s'}(\vec{x},t)\psi^{(\alpha)}_{\vec{p},s}(\vec{x},t) = \delta_{\alpha'\alpha}\delta_{s's} \tag{4.18}$$

5 Dirac γ-matrices

Instead of $(\beta, \vec{\alpha})$, one can also use

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma} \equiv \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \qquad \gamma^\mu \equiv \left(\gamma^0, \vec{\gamma}\right), \;\; \gamma_\mu \equiv \left(\gamma^0, -\vec{\gamma}\right) .$$

Then the Dirac equation (4.1) can be expressed as (remember: $x^0 = ct$)

$$\left(i\hbar\gamma^0 \frac{\partial}{\partial x^0} - \vec{\gamma}\cdot\hat{\vec{p}} - mc\right)\psi(\vec{x}, t) = 0 \tag{5.1}$$

where $\hat{\vec{p}} = -i\hbar\vec{\nabla}$ is the momentum operator.

Now define the operator $\hat{p}^\mu$ in terms of the contravariant derivative (see No. 1) as

$$\hat{p}^\mu \equiv i\hbar\partial^\mu = i\hbar\left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right) = \left(i\hbar\frac{\partial}{\partial x^0}, \hat{\vec{p}}\right) \tag{5.2}$$

Then (5.1) becomes

$$\left(\hat{p}^\mu\gamma_\mu - mc\right)\psi(x) = 0 \tag{5.3}$$

Finally, we define the "slash notation": For any 4-vector V^μ, the 4×4 matrix $\not{V}$ is defined by

$$\not{V} \equiv \gamma_\mu V^\mu = \gamma^0 V^0 - \vec{\gamma}\cdot\vec{V} \tag{5.4}$$

Then we can express (5.3) as

$$\left(\hat{\not{p}} - mc\right)\psi(x) = 0 \tag{5.5}$$

The previous relations for the matrices $(\beta, \vec{\alpha})$ become

$$\begin{aligned} \left(\gamma^0\right)^2 &= 1\,, \qquad \left(\gamma^i\right)^2 = -1\,, \qquad (i = 1, 2, 3 \text{ fix}) \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \end{aligned} \tag{5.6}$$

where $\{A, B\} = AB + BA$ is the anticommutator. Instead of $\psi^\dagger$, it is convenient to use $\overline{\psi}$, which is defined by

$$\overline{\psi} \equiv \psi^\dagger\gamma^0$$

Then the probability density $\rho = \psi^\dagger\psi$ and the current $\vec{j} = c\,\psi^\dagger\vec{\alpha}\psi$ can be combined into a 4-vector

$$j^\mu = c\,\overline{\psi}\gamma^\mu\psi = \left(c\,\rho, \vec{j}\right)\,, \qquad (\text{current} \quad \text{conservation}: \;\; \partial_\mu j^\mu = 0) \tag{5.7}$$