6 Invariance of D.E. under coordinate transformations

Transformation of coordinates x^μ and 4-momentum operator $\hat{p}^\mu = i\hbar\partial^\mu$ from a system S to another system S':

$$x'^{\mu} = \Lambda^\mu{}_\nu\, x^\nu\,, \qquad \hat{p}'^{\mu} = \Lambda^\mu{}_\nu\, \hat{p}^\nu \tag{6.1}$$

Here $\Lambda^\mu{}_\nu$ is a "generalized" Lorentz transformation, including also rotations. For a pure rotation: $x^{0'} = x^0$, $x'^{i} = R^i{}_j\, x^j$, where $R^i{}_j$ is a 3×3 rotation matrix:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 0 & 0 \\ 0 & R^i{}_j \end{pmatrix} \tag{6.2}$$

Basic requirement: The Dirac equation must keep its basic form under coordinate transformations!

$$S: \qquad (\hat{\not{p}} - mc)\,\psi(x) = 0 \tag{6.3}$$

$$S': \qquad (\hat{\not{p}}' - mc)\,\psi'(x') = 0 \tag{6.4}$$

Define a 4×4 matrix $\hat{S}$ such that

$$\psi'(x') = \hat{S}\,\psi(x) \tag{6.5}$$

Then from Eq.(6.3) and (6.4) we obtain

$$\hat{S}^{-1}\,\hat{\not{p}}'\,\hat{S} = \hat{\not{p}} \tag{6.6}$$

From this, we derive a relation to determine $\hat{S}$ as follows: Using $\hat{\not{p}}' = \gamma^\sigma\, p'_\sigma$ and Eq. (6.1),

$$\begin{aligned} \left(\hat{S}^{-1}\,\gamma^\sigma\,\hat{S}\right)\,\Lambda_\sigma{}^\nu \hat{p}_\nu &= \gamma^\nu\,\hat{p}_\nu \\ \Rightarrow \left(\hat{S}^{-1}\,\gamma^\sigma\,\hat{S}\right)\,\Lambda_\sigma{}^\nu &= \gamma^\nu \\ \Rightarrow \left(\hat{S}^{-1}\,\gamma^\sigma\,\hat{S}\right)\,\Lambda_\sigma{}^\nu\,\Lambda^\mu{}_\nu &= \Lambda^\mu{}_\nu\,\gamma^\nu \end{aligned}$$

Using here the basic property of the matrix Λ (see Chapt.1, Eq.(1.8))

$$\Lambda_\sigma{}^\nu\,\Lambda^\mu{}_\nu = g_\sigma{}^\mu$$

we obtain finally

$$\left(\hat{S}^{-1}\,\gamma^\mu\,\hat{S}\right) = \Lambda^\mu{}_\nu\,\gamma^\nu \tag{6.7}$$

Note: This relation tells that the "matrix transformation" of γ^μ (l.h.s. of (6.7)) is the same as the "4-vector transformation" of γ^μ (r.h.s. of (6.7)).

Take the dagger (†) of (6.7), and use

$$\gamma^{\mu\dagger} = \gamma^0\,\gamma^\mu\,\gamma^0 = \left(\gamma^0, -\vec{\gamma}\right)$$

Then we obtain from (6.7)

$$\hat{S}^\dagger\,\gamma^0\gamma^\mu\,\gamma^0\,\left(\hat{S}^{-1}\right)^\dagger = \Lambda^\mu{}_\nu\,\gamma^0\,\gamma^\nu\,\gamma^0$$

Multiply this from left and right by γ^0 to get

$$\left(\gamma^0\,\hat{S}^\dagger\,\gamma^0\right)\gamma^\mu\,\left(\gamma^0\,\left(\hat{S}^{-1}\right)^\dagger\,\gamma^0\right) = \Lambda^\mu{}_\nu\,\gamma^\nu$$

Comparing this with Eq.(6.7) we obtain the relation

$$\gamma^0\,\hat{S}^\dagger\,\gamma^0 = \hat{S}^{-1} \tag{6.8}$$

We will discuss the form of the matrix $\hat{S}$ later.

A similar discussion is possible also for parity transformation:

$$x^\mu \longrightarrow x'^\mu = \left(x^0, -\vec{x}\right) = x_\mu = g_{\mu\nu}x^\nu$$

Comparing with (6.1), we see that for parity transformations we can substitute $\Lambda^\mu{}_\nu \longrightarrow g_{\mu\nu}$. If we define a 4×4 matrix $\hat{P}$ like (6.5) by

$$\psi'(x') = \hat{P}\,\psi(x)$$

then we can use (6.7) to get

$$\left(\hat{P}^{-1}\,\gamma^\mu\,\hat{P}\right) = g_{\mu\nu}\,\gamma^\nu = \gamma_\mu = \left(\gamma^0, -\vec{\gamma}\right) \tag{6.9}$$

This is satisfied for

$$\hat{P} = \gamma^0$$

Eq.(6.8) is also satisfied, because $\hat{P} = \hat{P}^\dagger = \hat{P}^{-1}$.

7 Bilinear forms

A "bilinear form" is defined as

$$j(x) \equiv \overline{\psi}(x)\,\Gamma\,\psi(x) \tag{7.1}$$

where Γ is a Dirac matrix.

The following 16 Dirac matrices often appear in applications:

$$\Gamma = \left(1,\ \gamma^\mu,\ \sigma^{\mu\nu},\ \gamma^5,\ \gamma^\mu\gamma_5\right) \tag{7.2}$$

Here $\sigma^{\mu\nu}$ is defined by the commutator of two γ-matrices:

$$\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu] \tag{7.3}$$

and $\gamma_5 = \gamma^5$ is defined by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.4}$$

From these relations one can show, for example, that

$$\{\gamma^\mu, \gamma_5\} = 0\,, \qquad [\sigma^{\mu\nu}, \gamma_5] = 0\,, \qquad \left[\hat{S}, \gamma_5\right] = 0$$

Using then relations (6.7) and (6.9), one can show the properties of the bilinear forms (7.1) under Lorentz transformations, rotations, and parity transformations.

For example, for $\Gamma = c\,\gamma^\mu$, the bilinear (7.1) is the probability current $j^\mu(x) = c\overline{\psi}(x)\,\gamma^\mu\,\psi(x)$. After a Lorentz transformation it becomes

$$\begin{aligned} j'^{\mu}(x') &= c\,\psi'^{\dagger}(x')\gamma^0\gamma^\mu\psi'(x') \\ &= c\psi^\dagger(x)\,\hat{S}^\dagger\,\gamma^0\,\gamma^\mu\,\hat{S}\,\psi(x) \\ &= c\,\psi^\dagger(x)\,\gamma_0\,\hat{S}^{-1}\,\gamma^\mu\,\hat{S}\,\psi(x) \\ &= c\,\Lambda^\mu{}_\nu\,\overline{\psi}(x)\,\gamma^\nu\,\psi(x) = c\,\Lambda^\mu{}_\nu\,j^\nu(x) \end{aligned}$$

Therefore the current $j^\mu(x)$ behaves as a 4-vector under Lorentz transformations. Under pure rotations, the density $j^0(x)$ is invariant, and $\vec{j}(x)$ behaves a 3-vector. Under parity transformations, $j^0(x)$ is invariant, and $\vec{j}(x)$ changes the sign.

By similar calculations, one can obtain also the results for the behavior of other bilinears under Lorentz transformations (LT) and parity transformations (PT). If we define $(-1)^\mu \equiv 1$ for $\mu = 0$ and $(-1)^\mu \equiv -1$ for $\mu = 1, 2, 3$ we obtain

- $\overline{\psi}\psi$: Scalar under LT, scalar under PT $\Rightarrow$ This is called a "scalar".
- $\overline{\psi}\gamma^\mu\psi$: 4-vector under LT, $(-1)^\mu$ under PT $\Rightarrow$ This is called a "vector".
- $\overline{\psi}\gamma_5\psi$: Scalar under LT, (-1) under PT $\Rightarrow$ This is called a "pseudo-scalar".
- $\overline{\psi}\gamma^\mu\gamma_5\psi$: 4-vector under LT, $-(-1)^\mu$ under PT $\Rightarrow$ This is called a "pseudo-vector".
- $\overline{\psi}\sigma^{\mu\nu}\psi$: Tensor (rank 2) under LT, $(-1)^\mu(-1)^\nu$ under PT $\Rightarrow$ This is called a (second rank) "tensor".

Note: For free particles with momentum $\vec{p}$ and positive energy E_p we have
$\overline{\psi}(x)\Gamma\psi(x) = \frac{1}{V}\frac{mc^2}{E_p}\left(\overline{u}(\vec{p},s)\Gamma u(\vec{p},s)\right)$.
Consider the case $\Gamma = \gamma^\mu\gamma_5$. For $\mu = i$ this is related to the spin operator (in units of $\hbar$) $\hat{\vec{S}} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ by

$$\hat{\vec{S}} = \frac{1}{2}\gamma^0\,\vec{\gamma}\,\gamma_5$$

The following pseudo-vector can then be called the "spin 4-vector":

$$S^\mu(p) = \frac{1}{2}\overline{u}(\vec{p},s)\gamma^\mu\gamma_5\,u(\vec{p},s) \tag{7.5}$$

The reason for this is:

- In the rest system of the particle, $S^0(\vec{p}=0) = 0$ and

$$\vec{S}(\vec{p}=0) = \frac{1}{2}u^\dagger(\vec{p}=0,s)\,\vec{\Sigma}\,u(\vec{p}=0,s) = \frac{1}{2}\phi(s)^\dagger\,\vec{\sigma}\,\phi(s) = \frac{1}{2}\vec{n} \equiv \vec{S}_0 \tag{7.6}$$

  which gives the spin direction in the rest system[1]. Therefore, $S^\mu(\vec{p}=0) = \left(0, \vec{S}_0\right)$.

- The vector $S^\mu(p)$ is obtained from $S^\mu(\vec{p}=0)$ by a Lorentz transformation with velocity $\vec{v} = -\vec{p}c/E_p$:

$$S^\mu(p) = \Lambda^\mu{}_\nu(\vec{v} = -\frac{\vec{p}c}{E_p})\,S^\nu(\vec{p}=0) = \left(\frac{\vec{p}\cdot\vec{S}_0}{mc},\, \vec{S}_0 + \frac{\vec{p}\left(\vec{p}\cdot\vec{S}_0\right)}{m(E_p+mc^2)}\right) \tag{7.7}$$

Then the 3-vector $\vec{S}(p)$ is the "expectation value" of the spin operator between spinors, $\vec{S}(p) = \frac{1}{2}\,u^\dagger(\vec{p},s)\vec{\Sigma}\,u(\vec{p},s)$, and depends on the momentum ! (Remember that $[H,\Sigma_i] \neq 0$, and the spinors are eigenvectors of the Hamiltonian H but *not* of the spin operator.)

[1] Remember that the 2-component Pauli spinor $\phi(s)$ is an eigenvector of $\frac{1}{2}(\vec{\sigma}\cdot\vec{n})$ with eigenvalue $s = \pm 1/2$.