# 8 Spinor transformations

Here we discuss the form of the spinor transformation matrix $\hat{S}$, which is defined by (see Sect. 6)

$$\left(\hat{S}^{-1}\,\gamma^\mu\,\hat{S}\right) = \Lambda^\mu{}_\nu\,\gamma^\nu \tag{8.1}$$

1. Lorentz transformations: We will show that

$$\hat{S} = e^{-\frac{1}{2}\vec{\omega}\cdot\vec{\alpha}} \tag{8.2}$$

satisfies (8.1). Here the vector $\vec{\omega}$ is in the direction of the velocity $\vec{v}$, and its form will be given below. $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ are the Dirac matrices.

- Infinitesimal transformation: Setting $\hat{S} \to \hat{s}$, $\Lambda \to \lambda$, Eq.(8.1) becomes

$$\left(\hat{s}^{-1}\,\gamma^\mu\,\hat{s}\right) = \lambda^\mu{}_\nu\,\gamma^\nu \tag{8.3}$$

On the l.h.s. we can use

$$\hat{s} = 1 - \frac{1}{2}\vec{\omega}\cdot\vec{\alpha} \tag{8.4}$$

Then the l.h.s. of (8.3), up to $\mathcal{O}(\vec{\omega})$, is:

$$\hat{s}^{-1}\,\gamma^0\,\hat{s} = \gamma^0 - \vec{\omega}\cdot\vec{\gamma} \tag{8.5}$$

$$\hat{s}^{-1}\,\gamma^i\,\hat{s} = \gamma^i - \omega^i\gamma^0 \tag{8.6}$$

On the r.h.s. of (8.3), we need the infinitesimal form of the Lorentz transformation, $x'^\mu = \lambda^\mu{}_\nu\,x^\nu$, up to $\mathcal{O}(\vec{v})$:

$$x'^0 = x^0 - \frac{\vec{v}}{c}\cdot\vec{x} = \left(g^0{}_\nu + \frac{v_k}{c}\,g^k{}_\nu\right)\,x^\nu \equiv \lambda^0{}_\nu\,x^\nu \tag{8.7}$$

$$x'^i = x^i - \frac{v^i}{c}\,x^0 = \left(g^i{}_\nu - \frac{v^i}{c}\,g^0{}_\nu\right)\,x^\nu \equiv \lambda^i{}_\nu\,x^\nu \tag{8.8}$$

Therefore $\lambda^0{}_\nu\gamma^\nu$ and $\lambda^i{}_\nu\gamma^\nu$ agree with the r.h.s. of (8.5) and (8.6), if $\vec{\omega} = \vec{v}/c$ for the infinitesimal case. Then Eq.(8.3) holds for the infinitesimal case.

From (8.7) and (8.8) we have

$$\lambda^\mu{}_\nu = g^\mu{}_\nu + \frac{\omega_k}{n}\left(g^{\mu 0}\,g^k{}_\nu - g^{\mu k}\,g^0{}_\nu\right) \equiv \left(1 + \frac{\omega_k}{n}\,I^k\right)^\mu{}_\nu \tag{8.9}$$

where we used $\frac{\vec{\omega}}{n}$ instead of $\vec{\omega}$ ($n \to \infty$ will be taken at the end), and the $4 \times 4$ matrix $I^k$ is defined by

$$\left(I^k\right)^{\mu}{}_{\nu} = g^{\mu 0}\, g^{k}{}_{\nu} - g^{\mu k}\, g^{0}{}_{\nu} \tag{8.10}$$

If we take the coordinate system so that $\vec{\omega}$ (and $\vec{v}$) is along the $x$ axis, $\vec{\omega} = (\omega, 0, 0)$, then Eq. (8.9) becomes

$$\lambda^{\mu}{}_{\nu} = \left(1 - \frac{\omega}{n}\, I\right)^{\mu}{}_{\nu} \tag{8.11}$$

where the matrix $I$ is given by

$$I^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix has the properties

$$\left(I^2\right)^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad I^3 = I^5 = \cdots = I\,, \qquad I^4 = I^6 = \cdots = I^2$$

- Finite transformations: The finite spinor transformation $\hat{S}$ of (8.2) is obtained from the infinitesimal $\hat{s}$ by

$$\hat{S} = \lim_{n\to\infty}\, \hat{s}^n = \lim_{n\to\infty} \left(1 - \frac{1}{2}\frac{\vec{\omega}}{n} \cdot \vec{\alpha}\right)^n \tag{8.12}$$

Using then (8.3) $n$ times, we get ($\lim_{n\to\infty}$ in the formula below)

$$\begin{aligned} \hat{S}^{-1}\gamma^{\mu}\,\hat{S} &= \left(\hat{s}^{-1}\right)^n \gamma^{\mu}\,\hat{s}^n = \left(\hat{s}^{-1}\right)^{n-1} \left(\hat{s}^{-1}\,\gamma^{\mu}\,\hat{s}\right)\,\hat{s}^{n-1} \\ &= \lambda^{\mu}{}_{\nu}\,\left(\hat{s}^{-1}\right)^{n-2}\left(\hat{s}^{-1}\,\gamma^{\nu}\,\hat{s}\right)\,\hat{s}^{n-2} = \cdots = \left(\lambda^n\right)^{\mu}{}_{\nu}\,\gamma^{\nu} \end{aligned} \tag{8.13}$$

Therfore, if $\lim_{n\to\infty} \lambda^n$ becomes the usual Lorentz matrix $\Lambda^{\mu}{}_{\nu}$, Eq.(8.1) is satisfied. From (8.11) we obtain

$$\begin{aligned} (\lambda^n) &= \left(1 - \frac{\omega}{n}\, I\right)^n = e^{-\omega\, I} = \cosh\left(\omega\, I\right) - \sinh\left(\omega\, I\right) \\ &= 1 + I^2\left(\frac{\omega^2}{2!} + \frac{\omega^4}{4!} + \ldots\right) - I\left(\omega + \frac{\omega^3}{3!} + \ldots\right) \\ &= \left(1 - I^2\right) + I^2\left(1 + \frac{\omega^2}{2!} + \frac{\omega^4}{4!} + \ldots\right) - I\left(\omega + \frac{\omega^3}{3!} + \ldots\right) \\ &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + I^2\,\cosh\omega - I\,\sinh\omega = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

This agrees with the usual Lorentz matrix (for a tranformation along $x$ axis, $\vec{v} = (v, 0, 0)$) if

$$\sinh\omega = \gamma\,\frac{v}{c}\,, \qquad \cosh\omega = \gamma \tag{8.14}$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$. This concludes the check of Eq.(8.1) for Lorentz transformations.

2. Rotations: We will show that

$$\hat{S} = e^{-\frac{i}{2}\vec{\phi}\cdot\vec{\Sigma}} \tag{8.15}$$

satisfies[1]

$$\left(\hat{S}^{-1}\,\gamma_i\,\hat{S}\right) = R_{ij}\,\gamma_j \tag{8.16}$$

Here the vector $\vec{\phi}$ is in the direction of the rotation axis, and will be given below. $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \gamma_5\,\vec{\alpha}$ are the relativistic spin matrices. Because $\left[\vec{\Sigma}, \gamma^0\right] = 0$, Eq.(8.16) is the same as

$$\left(\hat{S}^{-1}\,\alpha_i\,\hat{S}\right) = R_{ij}\,\alpha_j \tag{8.17}$$

- Infinitesimal rotation: Setting $\hat{S} \to \hat{s}$, $R \to r$, Eq.(8.17) becomes

$$\left(\hat{s}^{-1}\,\alpha_i\,\hat{s}\right) = r_{ij}\,\alpha_j \tag{8.18}$$

On the l.h.s. we can use

$$\hat{s} = 1 - \frac{i}{2}\vec{\phi}\cdot\vec{\Sigma} \tag{8.19}$$

Then the l.h.s. of (8.18), up to $\mathcal{O}(\vec{\phi})$, is:

$$\hat{s}^{-1}\,\alpha_i\,\hat{s} = \alpha_i + \frac{i}{2}\phi_k\,\left[\Sigma_k, \alpha_i\right] = \alpha_i - \epsilon_{ijk}\,\phi_k\,\alpha_j \tag{8.20}$$

On the r.h.s. of (8.18), we need the infinitesimal form of the rotation matrix, $x'_i = r_{ij}\,x_j$, up to $\mathcal{O}(\vec{\varphi})$, where the direction of $\vec{\varphi}$ is the rotation axis and $\varphi$ is the rotation angle. From ordinary mechanics, this is given by

$$\vec{x}' \;=\; \vec{x} - \vec{x}\times\vec{\varphi} \Rightarrow x'_i = \left(\delta_{ij} - \epsilon_{ijk}\,\varphi_k\right)\,x_j \equiv r_{ij}\,x_j \tag{8.21}$$

[1]For the pure rotations, we will not distinguish between the 3-vectors $x^i$ and $x_i$.

Therefore $r_{ij}\,\alpha_j$ agrees with (8.20), if $\vec{\phi} = \vec{\varphi}$ for the infinitesimal case. Then Eq.(8.18) holds for the infinitesimal case.

If we take the coordinate system so that the rotation axis is the $z$ direction ($\vec{\phi} = (0, 0, \phi)$), then from (8.21) we have (with $\phi \to \phi/n$, and $n \to \infty$ at the end)

$$r_{ij} = \delta_{ij} - \epsilon_{3ij}\,\frac{\phi}{n} = \left(1 - \frac{\phi}{n}\,I\right)_{ij} \tag{8.22}$$

where the $3 \times 3$ matrix $I$ is defined by

$$I_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has the properties

$$\left(I^2\right)_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad I^3 = -I\,, \qquad I^4 = -I^2\,, \quad I^5 = I, \ldots$$

- <u>Finite rotations</u>: The finite rotation $\hat{S}$ of (8.15) is obtained from the infinitesimal $\hat{s}$ by

$$\hat{S} = \lim_{n\to\infty} \hat{s}^n = \lim_{n\to\infty} \left(1 - \frac{i}{2}\frac{\vec{\phi}}{n} \cdot \vec{\Sigma}\right)^n \tag{8.23}$$

Using then (8.18) $n$ times, we get ($\lim_{n\to\infty}$ in the formula below)

$$\begin{aligned} \hat{S}^{-1}\alpha_i\,\hat{S} &= \left(\hat{s}^{-1}\right)^n\,\alpha_i\,\hat{s}^n = \left(\hat{s}^{-1}\right)^{n-1}\,\left(\hat{s}^{-1}\,\alpha_i\,\hat{s}\right)\,\hat{s}^{n-1} \\ &= r_{ij}\,\left(\hat{s}^{-1}\right)^{n-2}\,\left(\hat{s}^{-1}\,\alpha_j\,\hat{s}\right)\,\hat{s}^{n-2} = \cdots = \left(r^n\right)_{ij}\,\alpha_j \end{aligned} \tag{8.24}$$

Therfore, if $\lim_{n\to\infty} r^n$ becomes the usual rotation matrix $R_{ij}$, Eq.(8.17) is satisfied. From (8.22) we obtain

$$\begin{aligned} \left(r^n\right) &= \left(1 - \frac{\phi}{n}\,I\right)^n = e^{-\phi\,I} = \cosh\left(\phi\,I\right) - \sinh\left(\phi\,I\right) \\ &= 1 + I^2\left(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \ldots\right) - I\left(\phi - \frac{\phi^3}{3!} + \ldots\right) \\ &= \left(1 + I^2\right) - I^2\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \ldots\right) - I\left(\phi - \frac{\phi^3}{3!} + \ldots\right) \\ &= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} - I^2\,\cos\phi - I\,\sin\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

This agrees with the usual rotation matrix (for a rotation about the $z$ axis, $\vec{\varphi} = (0, 0, \varphi)$) if $\phi = \varphi =$ angle of rotation. This concludes the check of Eq.(8.17) for rotations.