9 Projection operators for energy and spin

(i) Projection operators for positive and negative energy:

Remember from Sects. 3 and 4: The free Dirac equation has solutions with positive energy $E = E_p = \sqrt{(mc^2)^2 + \vec{p}^2 c^2}$ (spinor $w^{(+)}(\vec{p}, s) = u(\vec{p}, s)$) and negative energy $E = -E_p = -\sqrt{(mc^2)^2 + \vec{p}^2 c^2}$ (spinor $w^{(-)}(\vec{p}, s) = v(-\vec{p}, s)$).
Then the following matrices are projection operators for positive and negative energy:

$$\Lambda^{(+)}_{\alpha\beta}(\vec{p}) \equiv \sum_{s=\pm 1/2} u_\alpha(\vec{p}, s)\, \overline{u}_\beta(\vec{p}, s) \tag{9.1}$$

$$\Lambda^{(-)}_{\alpha\beta}(\vec{p}) \equiv -\sum_{s=\pm 1/2} v_\alpha(\vec{p}, s)\, \overline{v}_\beta(\vec{p}, s) \tag{9.2}$$

- From the completeness relation of the spinors (for the proof, see the Appendix at the end of this Section) $\sum_s (u_\alpha\, \overline{u}_\beta - v_\alpha\, \overline{v}_\beta) = \delta_{\alpha\beta}$ we obtain

  $$\Lambda^{(+)}_{\alpha\beta}(\vec{p}) + \Lambda^{(-)}_{\alpha\beta}(\vec{p}) = (1)_{\alpha\beta}$$

  and from the normalization $\overline{u}(\vec{p}, s')\, u(\vec{p}, s) = -\overline{v}(\vec{p}, s')\, v(\vec{p}, s) = 1$ and orthogonality $\overline{u}(\vec{p}, s')\, v(\vec{p}, s) = \overline{v}(\vec{p}, s')\, u(\vec{p}, s) = 0$ we obtain

  $$\begin{aligned} \left(\Lambda^{(+)}\right)^2 &= \Lambda^{(+)}, \qquad \left(\Lambda^{(-)}\right)^2 = \Lambda^{(-)} \\ \Lambda^{(+)}\,\Lambda^{(-)} &= \Lambda^{(-)}\,\Lambda^{(+)} = 0 \end{aligned}$$

  which are the properties of projection operators. We also obtain

  $$\begin{aligned} \Lambda^{(+)}\, u &= u, \qquad \Lambda^{(+)}\, v = 0 \\ \Lambda^{(-)}\, u &= 0, \qquad \Lambda^{(-)}\, v = v \end{aligned}$$

  which confirms that $\Lambda^{(+)}$ projects onto the positive energy spinors u, and $\Lambda^{(-)}$ projects onto the negative energy spinors v.

- The explicit forms of the Dirac matrices $\Lambda^{(+)}$ and $\Lambda^{(-)}$ can be obtained from the Dirac equation. The basic equation is $(\not{p} - mc)\, w(\vec{p}, s) = 0$, or separately for positive and negative energies:

  $$\left(\frac{1}{c} E_p \gamma^0 - \vec{p} \cdot \vec{\gamma} - m\, c\right) u(\vec{p}, s) = 0 \tag{9.3}$$

  $$\left(\frac{1}{c} E_p \gamma^0 - \vec{p} \cdot \vec{\gamma} + m\, c\right) v(\vec{p}, s) = 0 \tag{9.4}$$

By comparing (9.3) with $\Lambda^{(-)}u = 0$, we see that $\Lambda^{(-)} \propto \left(\frac{1}{c}E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} - m\,c\right)$. By comparing (9.4) with $\Lambda^{(+)}v = 0$, we see that and $\Lambda^{(+)} \propto \left(\frac{1}{c}E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m\,c\right)$. By using $\Lambda^{(+)} + \Lambda^{(-)} = 1$, we finally obtain

$$\Lambda^{(+)}(\vec{p}) = \frac{1}{2m\,c}\left(\frac{1}{c}E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m\,c\right) = \frac{1}{2m\,c}(\not{p} + mc) \tag{9.5}$$

$$\Lambda^{(-)}(\vec{p}) = \frac{-1}{2m\,c}\left(\frac{1}{c}E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} - m\,c\right) = \frac{-1}{2m\,c}(\not{p} - m\,c) \tag{9.6}$$

where $p^0 \equiv E_p/c$ in the last forms of (9.5) and (9.6).

(ii) Projection operators for spin directions $s = \pm 1/2$:

Remember from Sect.7: The "spin 4-vector" is given by

$$S^\mu(\vec{p}, s) = \frac{1}{2}\overline{u}(\vec{p}, s)\gamma^\mu\gamma_5\,u(\vec{p}, s) \equiv s\,n^\mu(\vec{p}) \tag{9.7}$$

$$= s\left(\frac{\vec{p}\cdot\vec{n}_0}{mc}, \vec{n}_0 + \frac{\vec{p}\,(\vec{p}\cdot\vec{n}_0)}{m(E_p + mc^2)}\right) \tag{9.8}$$

$$\stackrel{\text{rest system}}{\longrightarrow} (0, s\,\vec{n}_0) \qquad \text{where} \quad s\,\vec{n}_0 \equiv \frac{1}{2}\left(\phi(s)^\dagger\,\vec{\sigma}\,\phi(s)\right)$$

Here the unit vector $\vec{n}_0$ is the spin quantization axis, i.e., the spin direction in the rest system is given by $\vec{S}_0 = s\vec{n}_0$ (where $s = \pm 1/2$). Note that the 4-vector $n^\mu(\vec{p})$ does not depend on s. The spinor $\phi(s)$ is a 2-component Pauli-spinor, chosen as an eigenvector of $\frac{1}{2}\vec{\sigma}\cdot\vec{n}_0$.

Because $n_\mu\,n^\mu$ is Lorentz invariant, with the value $n_\mu\,n^\mu = -1$ (clear in the rest system!), we obtain, by multiplying (9.7) by n_μ,

$$\frac{1}{2}\overline{u}(\vec{p}, s)\gamma_5\,\not{n}(\vec{p})\,u(\vec{p}, s) = s \qquad (\text{where} \quad s = \pm\frac{1}{2}) \tag{9.9}$$

Here we show that, more generally, the spinor $u(\vec{p}, s)$ satisfies the following eigenvalue equation:

$$\left(\frac{1}{2}\gamma_5\,\not{n}(\vec{p})\right)u(\vec{p}, s) = s\,u(\vec{p}, s) \tag{9.10}$$

Therefore the operator $\left(\frac{1}{2}\gamma_5\,\not{n}\right)$ is the relativistic generalization of $\frac{1}{2}\,(\vec{\sigma}\cdot\vec{n}_0)$, and the quantum number s is its eigenvalue. The operator $\left(\frac{1}{2}\gamma_5\,\not{n}\right)$ is called the "Pauli-Lubanski operator". For example, an electron beam polarized in the direction $\vec{n}_0$, as its comes out of an accelerator, is in an eigenstate of the Pauli-Lubanski operator.

Homework: Show that $[\gamma_5\,\not{n}\,,\,H]\;u(\vec{p}, s) = 0$, where H is the Dirac Hamiltonian, $\not{n} = n^\mu\gamma_\mu$ where n^μ is defined in Eq.(9.7) and (9.8), and γ_5 was defined in Sect. 7. Therefore, the Pauli-Lubanski operator does in general *not* commute with the Hamiltonian. Only if applied to the Dirac spinor u, the commutator "effectively vanishes", and $s = \pm\frac{1}{2}$ is a conserved quantum number.

Proof of (9.10):

We use the spinor Lorentz transformation matrix $\hat{S}$ of Sect. 6. Here we need the spinor tranformation from the rest system of the particle to the system where the particle has momentum $\vec{p}$ and velocity $\vec{v} = \vec{p}c/E_p$. For this purpose, we must choose $\vec{v} = -\vec{p}c/E_p$ as the velocity of the observer (velocity of the Lorentz transformation). Then we have the spinor transformation $u(\vec{p}, s) = \hat{S}(\vec{v})\, u(\vec{p} = 0, s)$, and on the l.h.s. of (9.10) we have

$$\gamma_5 \not{n}(\vec{p})\, u(\vec{p}, s) = \gamma_5\, \hat{S}(\vec{v}) \left(\hat{S}^{-1}(\vec{v}) \not{n}(\vec{p})\, \hat{S}(\vec{v})\right) u(\vec{p} = 0, s)$$

Use here the basic property of the matrix $\hat{S}$ (see Sect. 6, Eq.(6.7)) and of the Lorentz matrix (see Sect. 1):

$$\begin{aligned}
\hat{S}^{-1}(\vec{v}) \not{n}(\vec{p})\, \hat{S}(\vec{v}) &= \left(\hat{S}^{-1}(\vec{v})\, \gamma^\mu\, \hat{S}(\vec{v})\right) n_\mu(\vec{p}) = \left(\Lambda^\mu{}_\nu(\vec{v})\, \gamma^\nu\right) n_\mu(\vec{p}) \\
&= \Lambda_\nu{}^\mu(-\vec{v})\, \gamma^\nu\, n_\mu(\vec{p}) = \gamma^\nu\, n_\nu(\vec{p} = 0) = \not{n}(\vec{p} = 0)
\end{aligned}$$

By using the above relations, we obtain

$$\begin{aligned}
\frac{1}{2}\gamma_5 \not{n}(\vec{p})\, u(\vec{p}, s) &= \frac{1}{2}\gamma_5\, \hat{S}(\vec{v}) \not{n}(\vec{p} = 0)\, u(\vec{p} = 0, s) \\
&= -\frac{1}{2}\, \hat{S}(\vec{v}) \not{n}(\vec{p} = 0)\, \gamma_5\, u(\vec{p} = 0, s) = \frac{1}{2}\hat{S}(\vec{v})\, (\vec{\gamma}\gamma_5) \cdot \vec{n}\, u(\vec{p} = 0, s) \\
&= \frac{1}{2}\hat{S}(\vec{v})\, (\gamma_0 \vec{\gamma}\gamma_5) \cdot \vec{n}\, u(\vec{p} = 0, s) = \frac{1}{2}\, \hat{S}(\vec{v}) \left(\vec{\Sigma} \cdot \vec{n}\right) u(\vec{p} = 0, s) \\
&= s\, \hat{S}(\vec{v})\, u(\vec{p} = 0, s) = s\, u(\vec{p}, s)
\end{aligned}$$

which proofs Eq.(9.10). [Note: In the 4th equality above, we used that the rest spinor $u(\vec{p} = 0, s)$ has only upper components (lower components are zero), therefore $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts like a unit matrix.]

Using relation (9.10), we can define the spin projection operators $P_\pm$ as follows:

$$P_\pm = \frac{1}{2}\, (1 \pm \gamma_5 \not{n})$$

This satisfies the relations for projection operators:

$$P_+ + P_- = 1\,, \qquad P_+\, P_- = \frac{1}{4}\, (1 + \not{n}\not{n}) = \frac{1}{4}\left(1 + n^2\right) = 0$$

because of $n^2 = -\vec{n}^2 = -1$. If $P_\pm$ acts on the spinor $u(\vec{p}, s)$ we have

$$P_\pm\, u(\vec{p}, s) = \left(\frac{1}{2} \pm s\right) u(\vec{p}, s)$$

and therefore $P_+ u(\vec{p}, s = \frac{1}{2}) = u(\vec{p}, s = \frac{1}{2})$, $P_- u(\vec{p}, s = \frac{1}{2}) = 0$, and $P_- u(\vec{p}, s = -\frac{1}{2}) = u(\vec{p}, s = -\frac{1}{2})$, $P_- u(\vec{p}, s = \frac{1}{2}) = 0$.

Because of $S^\mu(\vec{p}, s) = s\, n^\mu(\vec{p})$ (see Eq.(9.7)), the projection operators can also be expressed as $P_+ \equiv P(s = \frac{1}{2})$ and $P_- \equiv P(s = -\frac{1}{2})$, where

$$P(s) = \frac{1}{2} + \gamma_5\, s \not{n} = \frac{1}{2} + \gamma_5 \not{S}$$

Then we have the more compact relation

$$P(s)\, u(\vec{p}, s') = \delta_{ss'}\, u(\vec{p}, s)$$

(iii) Projection operator for positive energy and spin $s = \pm 1/2$: This is the following operator (see Eq.(9.1)):

$$\Lambda^{(+)}_{\alpha\beta}(\vec{p}, s) = u_\alpha(\vec{p}, s)\ \overline{u}_\beta(\vec{p}, s)$$

The explicit form of this operator is obtained by multiplying the positive energy projection operator (9.1) by the spin projection operator $P(s)$. Using the explicit form of the positive energy projection operator, given by (9.5), and the spin projection operator $P(s)$ given above, we obtain

$$u_\alpha(\vec{p}, s)\ \overline{u}_\beta(\vec{p}, s) = \frac{1}{2m\,c}\left[(\not{p} + mc)\left(\frac{1}{2} + \gamma_5 \not{S}\right)\right]_{\alpha\beta}$$

This is a very useful relation, because one can express the expectation value of any Dirac matrix A between spinors by a trace:

$$\overline{u}(\vec{p}, s)\, A\, u(\vec{p}, s) = \mathrm{Tr}\left(u(\vec{p}, s)\overline{u}(\vec{p}, s) A\right) = \frac{1}{2m\,c}\mathrm{Tr}\left[(\not{p} + mc)\left(\frac{1}{2} + \gamma_5 \not{S}\right)\, A\right]$$

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Appendix: Proof of the completeness relation of spinors:

In the rest frame, the completeness is clear from the form of the spinors (see Sect. 3): Taking for example the z-axis as the spin quantization axis, then $u(\vec{p} = 0, s = \frac{1}{2}) = (1, 0, 0, 0)$, $u(\vec{p} = 0, s = -\frac{1}{2}) = (0, 1, 0, 0)$, $v(\vec{p} = 0, s = \frac{1}{2}) = (0, 0, 1, 0)$, $v(\vec{p} = 0, s = \frac{1}{2}) = (0, 0, 0, 1)$. Therefore,

$$\begin{aligned} \sum_s \Big(u_\alpha(\vec{p} = 0, s) u^\dagger_\beta(\vec{p} = 0, s) + v_\alpha(\vec{p} = 0, s) v^\dagger_\beta(\vec{p} = 0, s)\Big) &= \delta_{\alpha\beta} \\ \Rightarrow \sum_s \left(u_\alpha(\vec{p} = 0, s)\overline{u}_\beta(\vec{p} = 0, s) - v_\alpha(\vec{p} = 0, s)\overline{v}_\beta(\vec{p} = 0, s)\right) &= \delta_{\alpha\beta} \end{aligned}$$

where we used $\overline{v}(\vec{p} = 0, s) = v^\dagger(\vec{p} = 0, s)\gamma^0 = -v^\dagger(\vec{p} = 0, s)$. By using the Lorentz transformation of spinors as above (transformation velocity $\vec{v} = -\vec{p}c/E_p$) we obtain

$$\begin{aligned}
&\sum_s \left(u_\alpha(\vec{p},s)\overline{u}_\beta(\vec{p},s) - v_\alpha(\vec{p},s)\overline{v}_\beta(\vec{p},s)\right) \\
&\quad = \sum_s \hat{S}_{\alpha\alpha'}\, u_{\alpha'}(\vec{p}=0,s)\overline{u}_{\beta'}(\vec{p}=0,s)\, \hat{S}^{-1}_{\beta'\beta} - \hat{S}_{\alpha\alpha'}\, v_{\alpha'}(\vec{p}=0,s)\overline{v}_{\beta'}(\vec{p}=0,s)\, \hat{S}^{-1}_{\beta'\beta} \\
&\quad = \hat{S}_{\alpha\alpha'}\, \delta_{\alpha'\beta'}\, \hat{S}^{-1}_{\beta'\beta} = \delta_{\alpha\beta}
\end{aligned}$$

which is the completeness relation for the spinors used above.