6 S-matrix for scattering of electron in external field

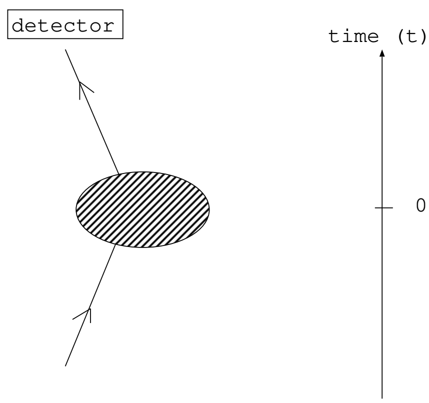
Remember the following formula for the time evolution of an electron wave function (positive energy) in an external electromagnetic field A^μ (see Sect. 14 of spring semester, Eqs.(14.5) and (14.6)):

$$i \int \mathrm{d}^3x' \; S_F(x-x')\,\gamma^0 \;\Psi_n(x') \;=\; \theta(t-t')\,\Psi_n(x) \tag{6.1}$$

$$i \int \mathrm{d}^3x \; \overline{\Psi}_n(x)\,\gamma^0\, S_F(x-x') \;=\; \theta(t-t')\,\overline{\Psi}_n(x') \tag{6.2}$$

Here S_F is the exact Feynman propagator of the electron in an external field, and $\Psi(x)$ is the exact wave function of the electron.

Consider the process of electron scattering by an external electromagnetic field, which may be created by another particle (for example, a nucleus):



Here the lines with arrows represent the electron, and the shaded area represents the space-time region of the interaction between the electron and the electromagnetic field. Suppose we have an exact wave function for the electron ($\Psi_i(x)$), which satisfies the following initial condition for time $t \to -\infty$:

$$\Psi_i(x) \overset{(t\to-\infty)}{\longrightarrow} \psi_i(x) \tag{6.3}$$

Here $\psi_i(x)$ is a free (positive energy) solution of the Dirac equation. Take the limit $t' \to -\infty$ on both sides of Eq.(6.1) for $n = i$:

$$\Psi_i(x) = i \lim_{t'\to-\infty} \int \mathrm{d}^3x'\, S_F(x-x')\,\gamma^0\,\psi_i(x')$$

Insert here the Dyson equation $S_F = S_{F0} + S_F(e\not{A})\,S_{F0}$ (see Sect. 14 of spring semester, Eq.(14.13)):

$$\Psi_i(x) = i \lim_{t'\to-\infty} \int \mathrm{d}^3x' \left[S_{F0}(x-x') + \int \mathrm{d}^4y\, S_F(x-y)(e\not{A}(y))\,S_{F0}(y-x') \right] \gamma^0\,\psi_i(x') \tag{6.4}$$

On the other hand, Eq.(6.1) for $t' \to -\infty$ and $A^\mu = 0$ (free electron) becomes

$$i \lim_{t'\to-\infty} \int d^3x'\, S_{F0}(x-x')\, \gamma^0\, \psi_i(x') = \psi_i(x)$$

Using this in Eq.(6.4) we get

$$\Psi_i(x) = \psi_i(x) + \int d^4y\, S_F(x-y)\, (e\not{A}(y))\, \psi_i(y) \tag{6.5}$$

At time $t \to +\infty$, place a detector which can filter out any free state $\psi_f(x)$ from the exact wave function (6.5). The probability amplitude to detect a particular state $\psi_f(x)$, which is contained in the wave function (6.5), is given by

$$S_{fi} \equiv \lim_{t\to\infty} \int d^3x\, \psi_f^\dagger(x)\, \Psi_i(x) \tag{6.6}$$

For all possible states (f, i) this is a matrix, which is called the <u>S-matrix</u>. Inserting here the formula (6.5) we obtain

$$\begin{aligned} S_{fi} &= \lim_{t\to\infty} \int d^3x\, \psi_f^\dagger(x) \left[\psi_i(x) + \int d^4y\, S_F(x-y)\, (e\not{A}(y))\, \psi_i(y)\right] \\ &= \delta_{fi} + \lim_{t\to\infty} \int d^3x \int d^4y\, \psi_f^\dagger(x)\, S_F(x-y)\, (e\not{A}(y))\, \psi_i(y) \end{aligned} \tag{6.7}$$

where we used the orthogonality of the free electron wave functions. Insert here the Dyson equation in the form (see Sect. 14 of spring semester, Eq.(14.10))

$$S_F(x-y) = S_{F0}(x-y) + \int d^4z\, S_{F0}(x-z)\, (e\not{A}(z))\, S_F(z-y)$$

and use Eq.(6.2) for $t \to \infty$ and $A^\mu = 0$ (free electron):

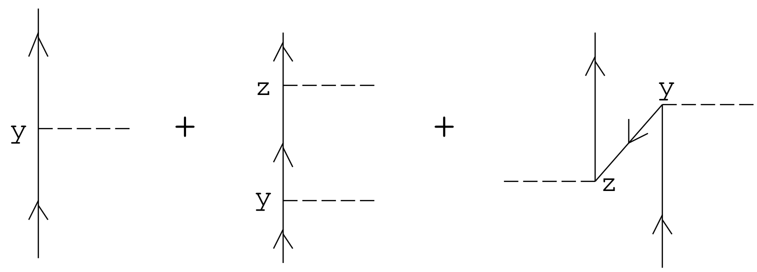
$$\lim_{t\to\infty} \int d^3x\, \psi_f^\dagger(x)\, S_{F0}(x-y) = -i\overline{\psi}_f(y)$$

Then, from (6.7), we obtain the following convenient form of the S-matrix:

$$S_{fi} = \delta_{fi} - i \int d^4y\, \overline{\psi}_f(y)(e\not{A}(y))\psi_i(y) - i \int d^4y \int d^4z\, \overline{\psi}_f(z)\, (e\not{A}(z))\, S_F(z-y)\, (e\not{A}(y))\psi_i(y) \tag{6.8}$$

Note that in (6.8) all wave functions are <u>free</u> wave functions, but $S_F(z-y)$ is the <u>exact</u> Feynman propagator, which can be expanded in perturbation theory according to the Dyson equation $S_F = S_{F0} + S_{F0}(e\not{A})S_{F0} + \ldots$.

Because the Feynman propagator $S_F(x-z)$ has two time orderings (see Sect. 13 of spring semester), the interaction terms in Eq.(6.8) can be graphically expressed as follows (up to second order perturbation theory):



Here time is running from bottom to top, and the dashed lines represent the (instantaneous) interactions of the electron with the external field (for example, the Coulomb potential of a nucleus).

Example: Scattering by a Coulomb potential (produced by a heavy nucleus).

$$\begin{aligned}
\psi_i(x) &= \sqrt{\frac{m}{E_p}}\,\frac{1}{\sqrt{V}}\,u(\vec{p},s)\,e^{-i\left(E_p y^0-\vec{p}\cdot\vec{y}\right)} \\
\psi_f(x) &= \sqrt{\frac{m}{E_{p'}}}\,\frac{1}{\sqrt{V}}\,u(\vec{p}',s')\,e^{-i\left(E_{p'} y^0-\vec{p}'\cdot\vec{y}\right)} \\
A^\mu(y) &= \left(A^0=-\frac{Ze}{4\pi|\vec{y}|},\ \vec{A}=0\right) \qquad (6.9)
\end{aligned}$$

Then the S-matrix (6.8), for $f\neq i$, to order e^2 becomes:

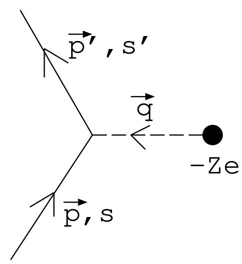
$$S_{fi}=iZ\frac{e^2}{4\pi}\frac{1}{V}\sqrt{\frac{m^2}{E_p\,E_{p'}}}\,\left(\overline{u}(\vec{p}',s')\,\gamma^0\,u(\vec{p},s)\right)\int\frac{\mathrm{d}^4y}{|\vec{y}|}\,e^{i(E_{p'}-E_p)y^0}\,e^{-i(\vec{p}'-\vec{p})\cdot\vec{y}}$$

Use here the relations

$$\begin{aligned}
\int_{-\infty}^{+\infty}\mathrm{d}y^0\,e^{i(E_{p'}-E_p)y^0} &= (2\pi)\,\delta(E_{p'}-E_p) \qquad (6.10)\\
\int\frac{\mathrm{d}^3y}{|\vec{y}|}\,e^{-i\vec{q}\cdot\vec{y}} &= \frac{4\pi}{\vec{q}^2} \qquad (\vec{q}=\vec{p}'-\vec{p})
\end{aligned}$$

Then we obtain

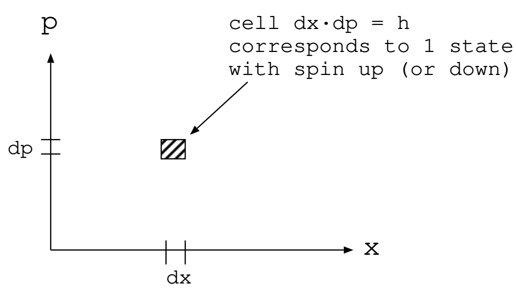
$$S_{fi}=\frac{iZe^2}{V}\,\frac{m}{E_p}\,\frac{\overline{u}(\vec{p}',s')\,\gamma^0\,u(\vec{p},s)}{\vec{q}^2}\,(2\pi)\,\delta(E_{p'}-E_p) \qquad (6.11)$$



The probability for electron scattering $i = (\vec{p}, s) \to f = (\vec{p}', s')$ is then given by $|S_{fi}|^2$. However, in a finite volume V, a state with sharp value of $\vec{p}'$ cannot be observed (because of the uncertainty relation). To calculate the probability for electron scattering, we need the number of final states in the volume V and in the momentum interval d^3p'. This number is called the phase space factor.

Consider first only 1 space dimension:

In classical physics, each point of the phase space (x, p) corresponds to one state. In quantum mechanics, because of the uncertainty principle $\mathrm{d}x\,\mathrm{d}p > h = 2\pi\hbar$, each cell of size h corresponds to one state:



In 3 space dimensions:

The number of states, with spin up or spin down, in the volume V and in the momentum interval d^3p', is equal to the number of cells in the phase space volume $(V \cdot \mathrm{d}^3p')$, and is given by

$$\frac{V\,\mathrm{d}^3p'}{(2\pi\hbar)^3} \tag{6.12}$$

Therefore, the probability for scattering into any of these $\frac{V\,\mathrm{d}^3p'}{(2\pi\hbar)^3}$ states is given by

$$|S_{fi}|^2\,\frac{V\,\mathrm{d}^3p'}{(2\pi\hbar)^3} \tag{6.13}$$

Definition of differential cross section:

$$\begin{aligned} \mathrm{d}^3\sigma &\equiv \frac{\text{number of particles scattered (per unit time) into } \mathrm{d}^3p'}{\text{number of incoming particles (per, time, per area)}} \\ &= N_{\text{in}} \left(|S_{fi}|^2 \cdot \frac{V\,\mathrm{d}^3p'}{(2\pi\hbar)^3} \frac{1}{\Delta T} \right) / |\vec{j}_{\text{in}}| \qquad (6.14) \end{aligned}$$

Here

- ΔT is the observation time $\simeq$ time it takes the electron to go from the accelerator to the detector. For a distance $L \simeq 1$ m, we have $\Delta T > 10^{-8}$ s;
- N_{in} is the number is incoming particles per unit time ;
- $\vec{j}_{\text{in}}$ is the incoming flux of particles:

$$\begin{aligned} \vec{j}_{\text{in}} &= \left(\overline{\psi}_i(x)\,\vec{\gamma}\,\psi_i(x)\right) \cdot N_{\text{in}} = \frac{m}{E_p}\frac{1}{V}\left(\overline{u}(\vec{p},s)\vec{\gamma}u(\vec{p},s)\right) \cdot N_{\text{in}} \\ &= \frac{\vec{p}}{E_p}\frac{N_{\text{in}}}{V} = \vec{v}\,\frac{N_{\text{in}}}{V} \end{aligned}$$

where $\vec{v}$ is the velocity of the incoming electrons.

Then we get the differential cross section as follows:

$$\begin{aligned} \mathrm{d}^3\sigma &= \frac{V}{v}|S_{fi}|^2 \cdot \frac{V\,\mathrm{d}^3p'}{(2\pi\hbar)^3}\frac{1}{\Delta T} \\ &= \frac{\mathrm{d}^3p'}{v(2\pi)^3}\frac{1}{\Delta T}\left(Ze^2\right)^2\left(\frac{m}{E_p}\right)^2 \frac{|\overline{u}(\vec{p}',s')\gamma^0 u(\vec{p},s)|^2}{\vec{q}^4}\left(2\pi\delta(E_{p'}-E_p)\right)^2 \qquad (6.15) \end{aligned}$$

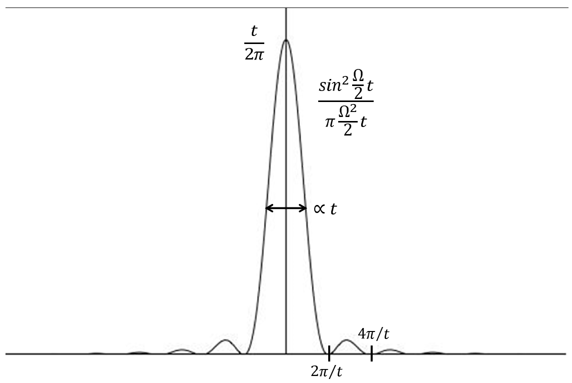
What is the meaning of the square of the delta function?

If the observation time is $\Delta T \simeq 10^{-8}$ s, then the integration over time (see Eq.(1.10)) should be replaced by $\int_{-\Delta T/2}^{+\Delta T/2} \mathrm{d}y^0$, and therefore

$$\begin{aligned} \left(2\pi\delta(E_{p'}-E_p)\right)^2 &\longrightarrow \left(\int_{-\Delta T/2}^{+\Delta T/2} \mathrm{d}t\, e^{i(E_{p'}-E_p)t}\right)^2 \\ &= 4\frac{\sin^2 \frac{E_{p'}-E_p}{2}\Delta T}{\left(E_{p'}-E_p\right)^2} = (2\pi\Delta T)\left(\frac{\sin^2\frac{\Delta E}{2}\Delta T}{\pi\frac{(\Delta E)^2}{2}\Delta T}\right) \end{aligned} \qquad (6.16)$$

Here $\Delta E \equiv E_{p'} - E_p$. Consider now the following expression as a function of ΔE:

$$\frac{\sin^2\frac{\Delta E}{2}\Delta T}{\pi\frac{(\Delta E)^2}{2}\Delta T} = \frac{\Delta T}{2\pi}\left(\frac{\sin\frac{\Delta E}{2}\Delta T}{\frac{\Delta E}{2}\Delta T}\right)^2$$



(In the figure, $t \equiv \Delta T$, and $\Omega \equiv \Delta E$.) Because $\Delta T \simeq 10^{-8}$ s is a "macroscopic time", we have [1] $\frac{2\pi\hbar}{\Delta T} \simeq 2\pi \times 10^{-7}$ eV, which is very small compared to a typical resolution energy of $\simeq 1$ eV. Therefore, $\frac{2\pi\hbar}{\Delta T}$ is practically zero, which means that the macroscopic time ΔT can be replaced by $\Delta T \to \infty$. Then we can use the relation

$$\frac{\sin^2 \frac{\Delta E}{2}\Delta T}{\pi\frac{(\Delta E)^2}{2}\Delta T} \overset{\Delta T\to\infty}{\longrightarrow} \delta(\Delta E)$$

Therefore, finally, we get for (6.16),

$$(2\pi\delta(E_{p'} - E_p))^2 \overset{\Delta T\to\infty}{\longrightarrow} (2\pi\Delta T)\ \delta(E_{p'} - E_p) \tag{6.17}$$

Note: A shorthand "derivation" of this result is simply as follows:

$$\begin{aligned}(2\pi\delta(E_{p'} - E_p))^2 &= (2\pi\delta(E_{p'} - E_p))\,(2\pi\delta(E_{p'} - E_p)) \\ &= (2\pi\delta(E_{p'} - E_p))\,(2\pi\delta(0)) \equiv (2\pi\delta(E_{p'} - E_p))\,\Delta T\end{aligned}$$

Physically, it means that if we observe the particle for a macroscopic time ΔT, there is no uncertainty of the energy. (Note that the static Coulomb potential cannot transfer energy, so we must get energy conservation: $E_p = E_{p'}$.)

Using the result (6.17) in the differential cross section (6.15), and $\mathrm{d}^3p' = p'^2\mathrm{d}p'\,\mathrm{d}\Omega'$ with $p = p'$ from the energy conserving delta function, we get (with $\alpha \equiv e^2/(4\pi) \simeq 1/137$)

$$\frac{\mathrm{d}^3\sigma}{\mathrm{d}p'\,\mathrm{d}\Omega'} = \frac{4Z^2\alpha^2p^2}{v}\left(\frac{m}{E_p}\right)^2 \delta(E_{p'} - E_p)\,\frac{|\overline{u}(\vec{p}', s')\gamma^0 u(\vec{p}, s)|^2}{\vec{q}^4}$$

[1] Note that $\hbar \simeq 10^{-15}$ eV $\cdot$ s.

We can integrate this over p', using $\delta(E_{p'} - E_p) = \frac{1}{v}\delta(p' - p)$ to get the usual "differential cross section":

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'} \equiv \frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega'} = \frac{4Z^2\alpha^2 m^2}{\vec{q}^4} \left|\overline{u}(\vec{p}', s')\gamma^0 u(\vec{p}, s)\right|^2 \tag{6.18}$$

Here the momentum transfer is given in terms of the scattering angle θ by

$$\vec{q}^2 = (\vec{p}' - \vec{p})^2 = 2p^2\,(1 - \cos\theta) = 4p^2\,\sin^2\frac{\theta}{2} \tag{6.19}$$

In the nonrelativistic limit we have $\overline{u}(\vec{p}', s')\gamma^0 u(\vec{p}, s) \simeq 1$, and Eq.(6.18) becomes the Rutherford cross section.

Relativistic effects from electron spin, Mott cross section:

Here we calculate the "unpolarized cross section":

- The initial electrons have equal probability for spin up ($s = 1/2$) and spin down ($s = -1/2$) $\Rightarrow$ average $\frac{1}{2}\sum_s$;
- Both spin directions of the final electron are observed, i.e., the detector does not differentiate between the spin directions
  $\Rightarrow$ sum $\sum_s$.

Then the unpolarized differential cross section becomes

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega'} = \frac{2Z^2\alpha^2 m^2}{\vec{q}^4} \sum_{ss'} \left|\overline{u}(\vec{p}', s')\gamma^0 u(\vec{p}, s)\right|^2 \tag{6.20}$$

We now use the following identity for the square of spinor matrix elements: If Γ is a Dirac γ-matrix, then

$$\begin{aligned}
|\overline{u}(f)\,\Gamma\,u(i)|^2 &= (\overline{u}(f)\,\Gamma\,u(i))\,\left(u^t(f)\gamma^0\,\Gamma^*\,u^*(i)\right) = (\overline{u}(f)\,\Gamma\,u(i))\,\left(u^\dagger(i)\,\Gamma^\dagger\,\gamma^0 u(f)\right) \\
&= (\overline{u}(f)\,\Gamma\,u(i))\,\left(\overline{u}(i)\left(\gamma^0\Gamma^\dagger\gamma^0\right)u(f)\right)
\end{aligned}$$

If we define $\overline{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0$, and indicate the Dirac indices $\alpha, \beta, \alpha', \beta'$ explicitly, we obtain

$$\begin{aligned}
\sum_{ss'} |\overline{u}(f)\,\Gamma\,u(i)|^2 &= \sum_{ss'} \overline{u}_\alpha(\vec{p}', s')\Gamma_{\alpha\beta}\,u_\beta(\vec{p}, s)\,\overline{u}_{\alpha'}(\vec{p}, s)\overline{\Gamma}_{\alpha'\beta'}\,u_{\beta'}(\vec{p}', s') \\
&= \mathrm{Tr}\,\left(\Lambda_+(\vec{p}')\,\Gamma\,\Lambda_+(\vec{p})\,\overline{\Gamma}\right)
\end{aligned}$$

where Tr means the trace over the Dirac indices, and the positive energy projection operator is given by (see Sect. 9 of spring semester, Eqs.(9.1) and (9.5))

$$\Lambda_+(\vec{p}) = \frac{\not{p} + m}{2m} \qquad \text{(here } \not{p} = E_p \gamma^0 - \vec{p}\cdot\vec{\gamma}\text{)}$$

In our case (Eq.(6.20)) we have $\Gamma = \overline{\Gamma} = \gamma^0$, and the unpolarized differential cross section becomes

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2}{2\vec{q}^4}\,\mathrm{Tr}\left[(\not{p}' + m)\,\gamma^0\;(\not{p} + m)\,\gamma^0\right] \tag{6.21}$$

There are many theorems about traces of products of γ-matrices $\Gamma = (\gamma^0, \gamma^i)$. Here we just need the following two theorems:

- The trace of a product of an odd number of Γ - matrices is zero:

$$\mathrm{Tr}\,(\Gamma_1 \dots \Gamma_n) = 0 \qquad \text{if } n = \text{odd} \tag{6.22}$$

  Proof: Using the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which satisfies $\gamma_5^2 = 1$ and $\{\gamma_5, \Gamma\} = 0$, and the property of the trace $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we have for n=odd

$$\begin{aligned} \mathrm{Tr}\,(\Gamma_1 \dots \Gamma_n) &= \mathrm{Tr}\,(\Gamma_1 \dots \Gamma_n \gamma_5\gamma_5) \\ &= \mathrm{Tr}\,(\gamma_5\Gamma_1 \dots \Gamma_n\gamma_5) = (-1)^n\;\mathrm{Tr}\,(\Gamma_1 \dots \Gamma_n) = -\mathrm{Tr}\,(\Gamma_1 \dots \Gamma_n) = 0 \end{aligned}$$

- For the product of two and four γ-matrices we have the following formulas:

$$\mathrm{Tr}\;(\gamma^\mu\gamma^\nu) = 4\,g^{\mu\nu} \tag{6.23}$$

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right) \tag{6.24}$$

  Proof of (6.23): Using $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and the anticommutation relations of γ-matrices, we have

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}\,\mathrm{Tr}\,1 = 4\,g^{\mu\nu}$$

  Eq.(6.24) can be derived by using similar methods.

  Note that the formula (6.23) gives the following result for any 4-vectors a, b:

$$\mathrm{Tr}\,(\not{a}\not{b}) = a_\mu b_\nu \mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4a_\mu b_\nu\, g^{\mu\nu} = 4a\cdot b$$

  In the same way, the formula (6.24) gives the following result for any 4-vectors a, b, c, d:

$$\mathrm{Tr}\,(\not{a}\not{b}\not{c}\not{d}) = 4\,(a\cdot b\;c\cdot d - a\cdot c\;b\cdot d + a\cdot d\;b\cdot c)$$

Now we can continue with the calculation of the unpolarized cross section (6.21): The trace factor becomes

$$\begin{aligned} \mathrm{Tr}\left[(\not{p}'+m)\,\gamma^0\,(\not{p}+m)\,\gamma^0\right] &= \mathrm{Tr}\left(m^2+\not{p}'\gamma^0\not{p}\gamma^0\right) \\ = 4\left(m^2+2E_p\,E_{p'}-p\cdot p'\right) & \qquad (6.25) \end{aligned}$$

Using now (with $E \equiv E_p = E_{p'}$)

$$p\cdot p' = E^2-\vec{p}^2\cos\theta = m^2+\vec{p}^2\,(1-\cos\theta) = m^2+2\vec{p}^2\sin^2\frac{\theta}{2}$$

we finally get for the trace factor (6.25)

$$\mathrm{Tr}\left[(\not{p}'+m)\,\gamma^0\,(\not{p}+m)\,\gamma^0\right] = 8E^2\left(1-\vec{v}^2\sin^2\frac{\theta}{2}\right)$$

Inserting this into the cross section (6.21), and using also the relation (6.19), we finally obtain

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2}{4p^2v^2\sin^4\frac{\theta}{2}}\left(1-v^2\sin^2\frac{\theta}{2}\right) \qquad (6.26)$$

Here we denote the magnitude squared of 3-vectors by $p^2 \equiv \vec{p}^2$ and $v^2 \equiv \vec{v}^2$. In our natural units, $\alpha = e^2/(4\pi) \simeq 1/137$ is the "fine structure constant".

The formula (6.26) is called the Mott cross section, and describes elastic scattering of electrons on a Coulomb potential (for example, produced by a heavy spinless nucleus).