7 Effects of finite size of target

The Mott cross section describes the elastic scattering of electrons on a Coulomb potential, i.e., on a heavy point nucleus with spin zero (no magnetic field). For a nucleus with finite size, the scalar potential $A^0(y)$ is modified as follows (see Sect.6, Eq.(6.9)):

$$A^0(y) = -\frac{Ze}{4\pi|\vec{y}|} \longrightarrow \frac{-Ze}{4\pi}\int \mathrm{d}^3y' \, \frac{\rho_e(\vec{y}')}{|\vec{y}-\vec{y}'|} \tag{7.1}$$

where $(-Ze) > 0$ is the charge of the nucleus, and the electric charge density $\rho_e(\vec{y})$ is normalized to 1:

$$\int \mathrm{d}^3y \, \rho_e(\vec{y}) = 1 \tag{7.2}$$

For a point nucleus we have $\rho_e(\vec{y}) = \delta^{(3)}(\vec{y})$. Then, in the calculation of the S-matrix, we need the Fourier transform of $A_0(y)$ (see Sect.6, Eq.(6.10)):

$$A_0(q) = \int \mathrm{d}^3y \, A_0(\vec{y})\, e^{-i\vec{q}\cdot\vec{y}} \;=\; \frac{-Ze}{4\pi}\int \mathrm{d}^3y \int \frac{\mathrm{d}^3y'}{|\vec{y}-\vec{y}'|}\,\rho_e(\vec{y}')\, e^{-i\vec{q}\cdot\vec{y}} \tag{7.3}$$

A variable transform $(\vec{y},\vec{y}') \to (\vec{z},\vec{y}')$ with $\vec{z} = \vec{y}-\vec{y}'$ gives

$$\begin{aligned} A_0(q) &= \frac{-Ze}{4\pi}\int \frac{\mathrm{d}^3z}{|\vec{z}|}\, e^{-i\vec{q}\cdot\vec{z}} \int \mathrm{d}^3y' \, \rho_e(\vec{y}')\, e^{-i\vec{q}\cdot\vec{y}'} \\ &= \frac{-Ze}{\vec{q}^2}\, F_e(\vec{q}^2) \end{aligned} \tag{7.4}$$

where the electric form factor of the nucleus is defined as

$$F_e(\vec{q}^2) = \int \mathrm{d}^3y \, \rho_e(\vec{y})\, e^{-i\vec{q}\cdot\vec{y}} \tag{7.5}$$

The electric form factor is the Fourier transform of the electric charge density. It is normalized as $F_e(\vec{q}^2 = 0) = 1$.

The cross section is then given by (see Sect.6, Eq.(6.26)):

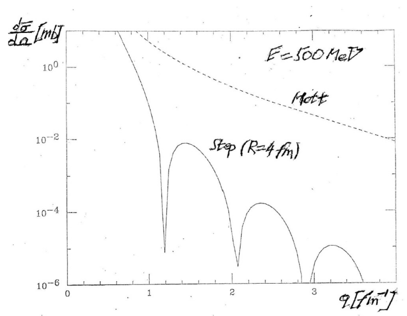
$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} |F_e(\vec{q}^2)|^2 \tag{7.6}$$

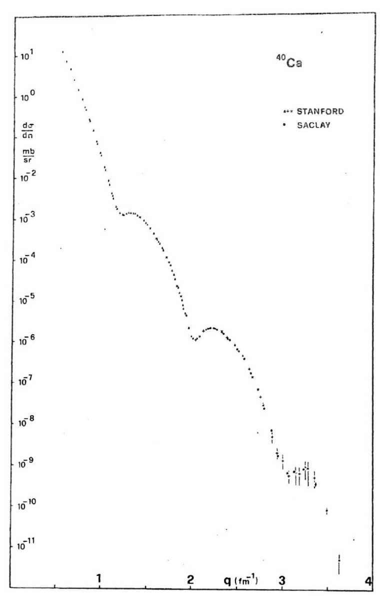
where $\left(\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}}$ is the "Mott cross section" for a point nucleus, given by Eq.(1.26) of No.12.
One can use the definition (7.5) to calculate the charge form factor for various forms of the charge density. Here we give some examples [1]:

[1] ρ_0 is a constant determined from the normalization (7.2). c and R are parameters determined by a fit to experimental data.

- Point function: $\rho_e(\vec{x}) = \delta^{(3)}(\vec{x}) \Rightarrow F_e(\vec{q}^2) = 1$.
- Exponential function: $\rho_e(\vec{x}) = \rho_0 e^{-r/c} \Rightarrow F_e(\vec{q}^2) = \frac{1}{(1+\vec{q}^2\, c^2)^2}$. ("Dipole form factor")
- Gauss function: $\rho_e(\vec{x}) = \rho_0\, e^{-\vec{x}^2/c^2} \Rightarrow F_e(\vec{q}^2) = e^{-\vec{q}^2 c^2/4}$.
- Step function: $\rho_e(\vec{x}) = \rho_0\, \theta(R - |\vec{x}|) \Rightarrow F_e(\vec{q}^2) = \frac{3}{(qR)^3}\,(\sin(qR) - qR\,\cos(qR))$.
  (Here $q \equiv |\vec{q}|$.)

As an example, we show the cross sections for the incident electron energy $E = 500$ MeV (=2.53 fm^{-1}). [2] Because this is large compared to the electron rest energy $m = 0.51$ MeV, we can use the "ultra-relativistic approximation" $E = \sqrt{m^2 + p^2} \simeq p$. The momentum transfer can then be expressed as $q = 2E\sin\frac{\theta}{2}$.
The left figure shows the cross section for a point target (Mott cross section), and for a step function with $R = 4$ fm. The right figure shows the experimental cross section for a ^{40}Ca target nucleus:





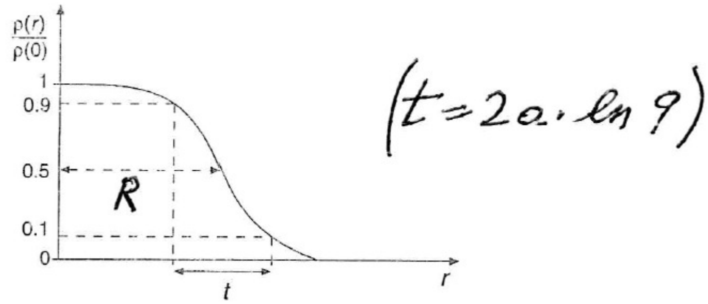
The experimental cross section for ^{40}Ca target is similar to the "step function" case. In general, the experimental data for a target nucleus with mass number A can be well described by the following "Fermi distribution":

$$\rho_e(r) = \frac{\rho_0}{1 + e^{-(r-R)/a}} \tag{7.7}$$

[2]Note: In our natural units, we have $\hbar c = 197$ MeV $\cdot$ fm $\equiv 1 \Rightarrow 1$ MeV $\equiv 1/197$ fm^{-1}. The cross sections are given in units of 1 mb $\equiv$ 1 millibarn $= 10^{-27}$ cm^2.

The parameters R and a are given for heavy nuclei (mass number $A > 40$) by:

$$R = r_0\, A^{1/3} \quad (r_0 = 1.1\,\mathrm{fm}), \qquad a = 0.54\,\mathrm{fm}$$



8 Heavy spin-1/2 target including finite size effects

Here we sketch how the results are modified if the target has spin 1/2. (We will discuss the case of spin-1/2 targets (nucleons) relativistically in later sections.)
In addition to the scalar potential A^0 (see Eq.(7.1)), the target also produces a vector potential:

$$\vec{A}(\vec{x}) = \frac{1}{4\pi} \int \mathrm{d}^3x' \, \frac{\vec{j}(\vec{x}')}{|\vec{x} - \vec{x}'|} \tag{8.1}$$

In our static approximation, the target does not move (no convection current), but it has a magnetic moment $\vec{m}$, which gives rise the the current

$$\vec{j}(\vec{x}) = -\left(\vec{m} \times \vec{\nabla}_x\right)\, \rho_m(\vec{x}) \tag{8.2}$$

where the magnetic moment density $\rho_m(\vec{x})$ is normalized by

$$\int \mathrm{d}^3x \, \rho_m(\vec{x}) = 1$$

Homework: By using partial integrations, show that Eq.(8.2) satisfies the usual definition of the magnetic moment in electrodynamics:

$$\vec{m} = \frac{1}{2} \int \mathrm{d}^3x \, \left(\vec{x} \times \vec{j}(\vec{x})\right)$$

The S-matrix, including the contribution from the vector potential, becomes (see Sect. 6, Eq.(6.8), (6.9)):

$$
\begin{aligned}
S_{fi} &= -ie \int \mathrm{d}^4x\, \overline{\psi}_f(x) \left[\gamma_0\, A^0(x) - \vec{\gamma}\cdot\vec{A}(x)\right] \psi_i(x) \\
&= -ie\frac{1}{V}\sqrt{\frac{m^2}{E_{p'}E_p}} \int \mathrm{d}^4x\, e^{i(E_{p'}-E_p)x^0}\, e^{-i\vec{q}\cdot\vec{x}} \left[\left(\overline{u}(\vec{p}',s')\gamma^0 u(\vec{p},s)\right) A^0(x) - \left(\overline{u}(\vec{p}',s')\vec{\gamma}u(\vec{p},s)\right)\cdot\vec{A}(x)\right] \\
&= -ie\frac{1}{V}\sqrt{\frac{m^2}{E_{p'}E_p}}\, \left(2\pi\delta(E_{p'}-E_p)\right) \left[\left(\overline{u}(\vec{p}',s')\gamma^0 u(\vec{p},s)\right) A^0(q) - \left(\overline{u}(\vec{p}',s')\vec{\gamma}u(\vec{p},s)\right)\cdot\vec{A}(q)\right] \qquad (8.3)
\end{aligned}
$$

The Fourier transform of the scalar potential was calculated in (7.4), and a similar calculation gives for the vector potential

$$\vec{A}(q) = \int \mathrm{d}^3x\, \vec{A}(\vec{x})\, e^{-i\vec{q}\cdot\vec{x}} = \frac{-i}{\vec{q}^2}\,(\vec{m}\times\vec{q})\; F_m(\vec{q}^2) \qquad (8.4)$$

where the magnetic form factor is defined as the Fourier transform of the magnetic moment density:

$$F_m(\vec{q}^2) = \int \mathrm{d}^3x\, e^{-i\vec{q}\cdot\vec{x}}\, \rho_m(\vec{x}) \qquad (8.5)$$

It is normalized as $F_m(\vec{q}^2 = 0) = 1$.

Homework: Use (8.1) and (8.2) to derive (8.4), using partial integrations.

Consider here the case of a nucleon target: Its magnetic moment is related to the spin vector $\vec{S}$ by

$$\vec{m} = \mu_N\, g_s\, \vec{S} \qquad (8.6)$$

where $\mu_N = -Ze/(2M)$ is the nuclear magneton ($Z = 1$), g_s is the spin g-factor of the nucleon, and $\vec{S}$ is the spin vector. The observed values are $g_s = 5.58$ for a proton, and $g_s = -3.82$ for a neutron [3]. The spin vector $\vec{S}$ for the case of electron-nucleon scattering is given by the transition matrix element of the spin operator $\hat{\vec{S}} = \vec{\sigma}/2$ between the Pauli spinors of the nucleon:

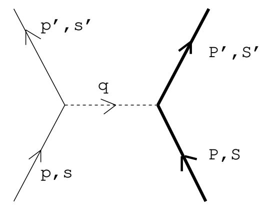
$$\vec{S} \equiv \phi^\dagger(S')\, \frac{\vec{\sigma}}{2}\, \phi(S) \qquad (8.7)$$

where $S, S' = \pm 1/2$ are the spin quantum numbers of the nucleon in the initial and final states.

Then the vector potential (8.4) takes the form

$$\vec{A}(q) = \frac{-Ze}{\vec{q}^2}\,\frac{-ig_s}{2M}\,\left(\vec{S}\times\vec{q}\right) F_m(\vec{q}^2)$$

[3] Remember that for a "Dirac particle" we have $g_s = 2$.



The unpolarized differential cross section becomes (see Sect. 6, Eq.(6.20)):

$$\frac{d\overline{\sigma}}{d\Omega} = \frac{Z^2\alpha^2 m^2}{\vec{q}^4} \sum_{ss'} \sum_{SS'} \mid \left(\overline{u}(\vec{p}', s')\gamma^0 u(\vec{p}, s)\right) F_e(\vec{q}^2)\delta_{SS'} + \frac{ig_s}{2M} \left(\overline{u}(\vec{p}', s')\vec{\gamma} u(\vec{p}, s)\right) \cdot \left(\vec{S} \times \vec{q}\right) F_m(\vec{q}^2)|^2 \tag{8.8}$$

One can show that the cross terms between the electric and magnetic contributions are zero. The summation over electron spins is performed by using traces over Dirac matrices (see Sect.6), and over nucleon spins by using traces of Pauli matrices. The result (which will be derived in the relativistic theory in later sections) is as follows:

$$\frac{d\overline{\sigma}}{d\Omega} = \left(\frac{d\overline{\sigma}}{d\Omega}\right)_{\text{Mott}} \left[\left(F_e(\vec{q}^2)\right)^2 + \frac{\vec{q}^2}{16M^2} g_s^2 \left(F_m(\vec{q}^2)\right)^2 \left(1 + 2\tan^2\frac{\theta}{2}\right)\right] \tag{8.9}$$

$$= \left(\frac{d\overline{\sigma}}{d\Omega}\right)_{\text{Mott}} \left[\left(G_E(\vec{q}^2)\right)^2 + b \left(G_M(\vec{q}^2)\right)^2 \left(1 + 2\tan^2\frac{\theta}{2}\right)\right] \tag{8.10}$$

In Eq.(8.10), we defined $b \equiv \vec{q}^2/(4M^2)$, and redefined the electric and magnetic form factors as follows:

$$G_E(\vec{q}^2) \equiv F_e(\vec{q}^2) \qquad (G_E(\vec{q}^2 = 0) = 1)$$

$$G_M(\vec{q}^2) \equiv \frac{g_s}{2} F_m(\vec{q}^2) \qquad (G_M(\vec{q}^2 = 0) = \frac{g_s}{2})$$

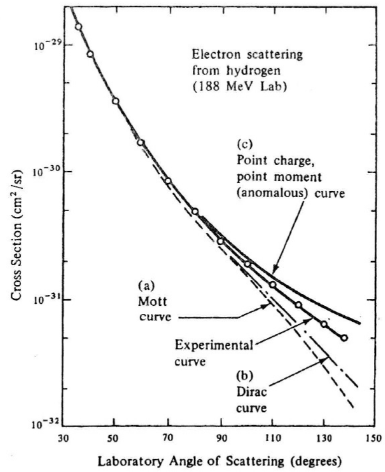
The fully relativistic formula for the cross section, which will be derived in later sections, is given by

$$\frac{d\overline{\sigma}}{d\Omega} = \left(\frac{d\overline{\sigma}}{d\Omega}\right)_{\text{Mott}} \left[\frac{\left(G_E(Q^2)\right)^2 + b\left(G_M(Q^2)\right)^2}{1+b} + 2b\left(G_M(Q^2)\right)^2 \tan^2\frac{\theta}{2}\right] \tag{8.11}$$

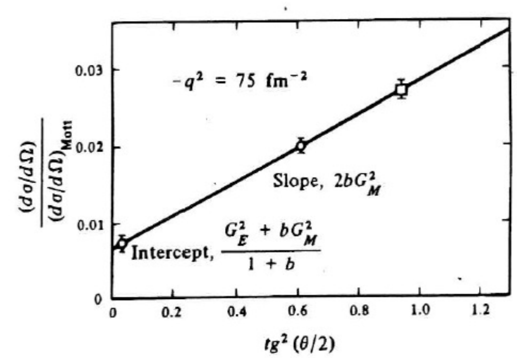
where $Q^2 = -q^2 = \vec{q}^2 - q_0^2 > 0$ is the 4-momentum transfer squared [4], and $b = Q^2/(4M^2)$. Eq.(8.11) is called the Rosenbluth formula.

[4] In electron scattering, the square of the 4-momentum transfer is always negative: $q^2 < 0$. Reason for this: If (k, k') are the electron momenta before and after scattering, then $q^2 = (k - k')^2 = 2\left(m^2 - k \cdot k'\right) = 2\left(m^2 - E_k E_{k'} + \vec{k} \cdot \vec{k}'\right)$, where we used the on-shell conditions $k'^2 = k^2 = m^2$. Because q^2 is Lorentz invariant, we can use the frame where $\vec{k} = 0$ $(E_k = m)$, and there evidently $q^2 < 0$. The 4-vector q^μ is therefore a "space-like" 4-vector.

Comparison of the cross section with experiment for incident electron energy $E = 188$ MeV:



(a) Mott curve means the Mott cross section (replace $G_E(Q^2) \to 1$ and $G_M(Q^2) \to 0$);
(b) Dirac curve means to replace $G_E(Q^2) \to 1$ (point charge) and $G_M(Q^2) \to 1$ (point magnetic moment with the Dirac value $g_s = 2$);
(c) Point charge, point moment (anomalous) curve means to replace $G_E(Q^2) \to 1$ (point charge) and $G_M(Q^2) \to g_s$ (point magnetic moment, $g_s = 5.58$ for proton and -3.82 for neutron).
The form factors $G_E(Q^2)$ and $G_M(Q^2)$ are extracted from the experimental cross section by using the Rosenbluth plot: For a fixed value of $Q^2 = 4E^2 \sin^2 \frac{\theta}{2}$, plot the cross section (8.11) as a function of $\tan^2 \frac{\theta}{2}$. The experimental data should lie on a straight line.



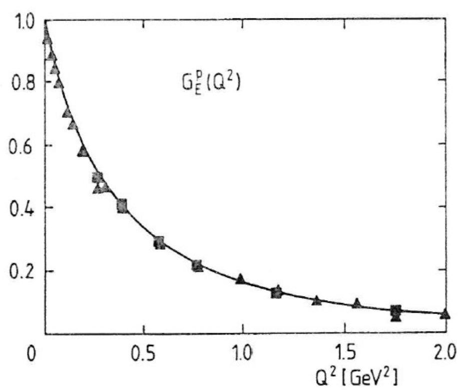
From the slope and the intercept of the straight line (which is fitted to experimental data), one obtains $G_E(Q^2)$ and $G_M(Q^2)$. [5]

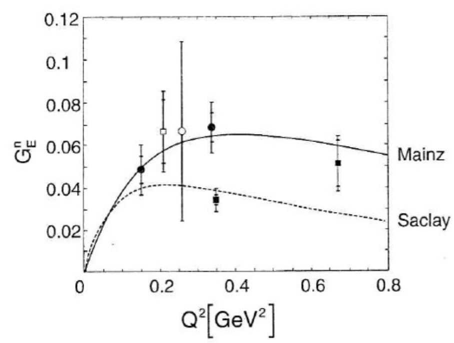
[5] In order to fix $Q^2 = 4E^2 \sin^2 \frac{\theta}{2}$ and vary θ, one has to vary also the incident electron energy E.

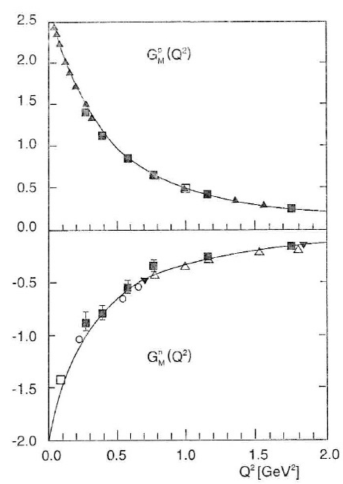
Observed form factors of the nucleon:

The nucleon form factors obtained from electron scattering experiments by using the Rosenbluth plot are shown in the figures below. The electric form factor of proton ($G_E^p(Q^2)$) and the magnetic form factors of proton and neutron ($G_M^p(Q^2)$ and $G_M^n(Q^2)$) are well described by the "dipole form factor" $G_D(Q^2) = \frac{1}{1+Q^2/\Lambda^2}$ with $\Lambda^2 = 0.71$ GeV^2:

$$G_E^p(Q^2) = G_D(Q^2)\,, \qquad G_M^p(Q^2) = \frac{g_s^p}{2}\, G_D(Q^2)\,, \qquad G_M^n(Q^2) = \frac{g_s^n}{2}\, G_D(Q^2)$$







In the nonrelativistic limit, the charge density is given by the Fourier transform of the charge form factor, and the magnetic moment density is given by the Fourier transform of the magnetic form factor. Then the charge density of proton, and the magnetic moment densities of proton and neutron are approximately exponential functions. (See Sect. 8, case "exponential function".) The charge density of neutron, obtained from a Fourier transform of G_E^n, is positive in the center (small r), and negative at the surface (large r). This can be explained by Yukawa theory, because the neutron has a cloud of virtual pions (see Figure below). Only charged pions contribute to the charge density, and in the case of a neutron only π^- is possible because of charge conservation.