10 Relativistic treatment of electron-proton scattering

Remember: The S-matrix (in lowest order perturbation theory) for the scattering of an electron in an external electromagnetic field $A^\mu(x)$ is given by (see Sect. 6, Eq.(6.8)):

$$S_{fi} = -ie \int \mathrm{d}^4x \, j^\mu_{fi}(x) \, A_\mu(x) \tag{10.1}$$

where $j^\mu_{fi}(x)$ is the electron "transition current" given by

$$j^\mu_{fi}(x) = \overline{\psi}_f(x)\gamma^\mu\psi_i(x) = e^{i(p'-p)x} \, j^\mu_{fi}(x=0) \tag{10.2}$$

with

$$j^\mu_{fi}(x=0) = \sqrt{\frac{m}{E_{p'}}}\sqrt{\frac{m}{E_{p'}}}\,\frac{1}{V}\,\overline{u}(\vec{p}', s')\gamma^\mu u(\vec{p}, s) \tag{10.3}$$

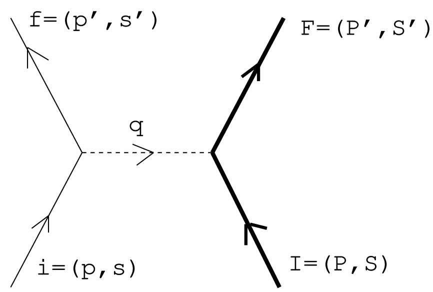

Figure 1: The initial hadron state I denotes a proton, and the final hadron state F denotes a proton (for elastic scattering, Bjorken $x = 1$) or a heavier state of hadrons ($p\pi^0$, $n\pi^+$, ...) for inelastic scattering (Bjorken $x < 1$).

The vector potential $A^\mu(x)$, which is produced by the hadron in the transition $I \to F$, is obtained from the Maxwell equation

$$\Box A^\mu(x) = e_p \, J^\mu_{FI}(x) \tag{10.4}$$

where $e_p = -e$ is the proton charge, and $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator. The solution to (10.4) is obtained as

$$A^\mu(x) = -e_p \int \mathrm{d}^4x' \, D(x-x') \, J^\mu_{FI}(x') \tag{10.5}$$

Here $D(x-x')$ is the Green function of the d'Alembert operator defined by

$$\Box_x \, D(x-x') = -\delta^{(4)}(x-x') \tag{10.6}$$

Using (10.5), the S-matrix (10.1) for electron-proton scattering becomes

$$S_{fi} = iee_p \int \mathrm{d}^4x \int \mathrm{d}^4x' \, j^{\mu}_{fi}(x) \, D(x-x') \, J_{\mu,FI}(x') \tag{10.7}$$

Because the proton initial state is described by a plane wave $e^{-iP\cdot x'}$ (independent of its internal structure), and the final hadron state (whatever it is) is also described by a plane wave $e^{iP'\cdot x'}$, the x' - dependence of the hadron transition current is of the form

$$J^{\mu}_{FI}(x') = e^{i(P'-P)\cdot x'} \, J^{\mu}_{FI}(x'=0) \tag{10.8}$$

Then we can perform the integrations over x and x' in (10.7) by

$$\begin{aligned} \int \mathrm{d}^4x \int \mathrm{d}^4x' \, e^{i(p'-p)\cdot x} \, e^{i(P'-P)\cdot x'} \, D(x-x') &\overset{(x,x')\to(z=x-x',x)}{=} \int \mathrm{d}^4x \int \mathrm{d}^4z \, e^{i(p'-p+P'-P)\cdot x} \, e^{-i(P'-P)\cdot z} \, D(z) \\ &= (2\pi)^4 \, \delta^{(4)}\left(P'+p'-P-p\right) \, D(q) \end{aligned} \tag{10.9}$$

where $q = p - p'$ is the momentum transfer (see Fig.1), and the Fourier transformed Green function $D(q)$ is obtained from (10.6) as

$$D(q) = \frac{1}{q^2} \tag{10.10}$$

Then the S-matrix (10.7) becomes

$$S_{fi} = \frac{iee_p}{q^2} \, (2\pi)^4 \, \delta^{(4)}\left(P'+p'-P-p\right) \, j^{\mu}_{fi} \, J_{\mu,FI} \tag{10.11}$$

where the transition currents are now defined at $x = 0$.

<u>Remember from Sect. 6</u>: In order to calculate the differential cross section from (10.11), we have to do the following:

- Calculate $|S_{fi}|^2$, by using the following rule:

$$\begin{aligned} \left[(2\pi)^4 \, \delta^{(4)}\left(P'+p'-P-p\right)\right]^2 &\equiv (2\pi)^4 \, \delta^{(4)}\left(P'+p'-P-p\right) \cdot (2\pi)^4 \, \delta^{(4)}\left(k=0\right) \\ &\equiv (2\pi)^4 \, \delta^{(4)}\left(P'+p'-P-p\right) \, V\Delta T \end{aligned}$$

  where V is the volume of the system, and ΔT is the observation time.

- Divide by the flux of incident particles. If we consider the laboratory frame (proton target at rest), this is the flux of incoming electrons (see Sect. 6):

$$|\vec{j}_{\text{in}}| = |\vec{v}| \, \frac{N_{\text{in}}}{V}$$

  where $\vec{v}$ is the velocity of the incoming electrons, and N_{in} is the number of incoming electrons.

- Multiply a factor

$$\frac{N_{\text{in}}}{\Delta T}\,\frac{V\mathrm{d}^3p'}{(2\pi)^3}\,\frac{V\mathrm{d}^3P'}{(2\pi)^3}$$

After these steps, we get for the differential cross section:

$$\mathrm{d}\sigma = \frac{e^4}{q^4}\,(2\pi)^4\,\delta^{(4)}\,(P'+p'-P-p)\,\frac{1}{v}\,\frac{\mathrm{d}^3p'}{(2\pi)^3}\,\frac{\mathrm{d}^3P'}{(2\pi)^3}\,|j^{\mu}_{fi}\,J_{\mu,FI}|^2 \tag{10.12}$$

To obtain this result, we have taken out the factors $\frac{1}{V}$, which are contained in the transition currents (see (10.3) for the electron):

$$j^{\mu}_{fi} \to \frac{1}{V}\,j^{\mu}_{fi}\,, \qquad J^{\mu}_{FI} \to \frac{1}{V}\,J^{\mu}_{FI}$$

where the transition currents j^{μ}_{fi} and J^{μ}_{FI} now have no factors $1/V$.
We will always assume that the spin of the electron in the final state (s') is not observed, and that the momentum and spins of the final hadrons ($\vec{P}'$ and S', where S' symbolically denotes the final spin of the hadrons) are also not observed. So, only the momentum of the electron in the final state $(\vec{p}')$ is observed by a detector. In this case, we have to sum over s' and S', and integrate over $\vec{P}'$ in Eq.(10.12). The factor $|j^{\mu}_{fi}\,J_{\mu,FI}|^2$ in (10.12) can be expressed as follows:

$$|j^{\mu}_{fi}\,J_{\mu,FI}|^2 = \left(j^{\mu}_{fi}\,j^{\nu *}_{fi}\right)\left(J^{\mu}_{FI}\,J^{\nu *}_{FI}\right) \tag{10.13}$$

We now define the leptonic tensor $\ell^{\mu\nu}$ and the hadronic tensor $W^{\mu\nu}$ by the following relations [1]:

$$\ell^{\mu\nu} \equiv 2m^2 \sum_{s'} \left(\overline{u}(\vec{p}',s')\gamma^\mu u(\vec{p},s)\right)\left(\overline{u}(\vec{p}',s')\gamma^\nu u(\vec{p},s)\right)^* \tag{10.15}$$

$$W^{\mu\nu} \equiv E_P \int \mathrm{d}^3P'\,\delta^{(4)}\left(P'+p'-P-p\right)\sum_{S'} J^\mu_{FI}\,J^{\nu *}_{FI} \tag{10.16}$$

Using also $\mathrm{d}^3p' = p'^2\mathrm{d}p'\,\mathrm{d}\Omega' = p' E_{p'}\,\mathrm{d}E_{p'}\,\mathrm{d}\Omega'$ we obtain the following important result:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_{p'}\,\mathrm{d}\Omega'} = \frac{e^4}{8\pi^2}\frac{p'}{p}\frac{1}{E_P\,q^4}\,\ell_{\mu\nu}W^{\mu\nu} \tag{10.17}$$

Remember that $q^2 = q_0^2 - \vec{q}^{\,2}$ is the square of the 4-momentum transfer in the scattering process, and in the laboratory system the initial proton 4-momentum is $P^\mu = \left(E_P = M, \vec{0}\right)$. Therefore the Bjorken variable $x = \frac{-q^2}{2P\cdot q} = \frac{2E_pE_{p'}\sin^2\theta}{(E_p - E_{p'})M}$ (see Sect. 9) can be measured by observing only the electron in the final state, i.e., the final electron energy ($E_{p'}$) and the electron scattering angle (θ). In the following, we will consider the following 2 processes:

- Elastic scattering: The final state (F) is a proton ($x = 1$).
- Inelastic inclusive scattering: The final state is not a proton ($x < 1$), but is not observed by a detector (i.e., only the final electron is observed, and only events with $x < 1$ are counted). In this case, we have to sum over all possible final hadronic states F.

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[1] The hadronic tensor (10.16) is defined in a general frame (not only the laboratory frame). It is often written in the form

$$\begin{aligned} W^{\mu\nu} &= \frac{2E_p}{2\pi}\int\frac{\mathrm{d}^3P'}{(2\pi)^3}\sum_{S'}(2\pi)^4\delta^{(4)}\left(P'+p'-P-p\right)\ J^\mu_{FI}\,J^{\nu *}_{FI} \equiv \frac{2E_P}{2\pi}\hat{\sum}_F (2\pi)^4\delta^{(4)}\left(P'+p'-P-p\right)\ J^\mu_{FI}\,J^{\nu *}_{FI} \\ &\equiv \frac{2E_P}{2\pi}\hat{\sum}_F (2\pi)^4\delta^{(4)}\left(P'+p'-P-p\right)\langle F|\hat{J}^\mu|I\rangle\langle F|\hat{J}^\nu|I\rangle^* \end{aligned} \tag{10.14}$$

where the sum over the final proton states is defined as $\hat{\sum}_F \equiv \int\frac{\mathrm{d}^3P'}{(2\pi)^3}\sum_{S'}$, and $\hat{J}^\mu$ is the current field operator. The factor $2E_p$ in the above expression of $W^{\mu\nu}$ reflects our non-covariant normalization of state vectors used so far in this course. The normalization and completeness relations in this convention are

$$\begin{aligned} \langle\vec{p}'|\vec{p}\rangle &= (2\pi)^3\delta^{(3)}(\vec{p}'-\vec{p}) \Leftrightarrow \langle\vec{p}|\vec{p}\rangle = V \\ \int\frac{\mathrm{d}^3p}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}| &= 1 \end{aligned}$$

In many texts, covariant normalization (c) is used, with $|\vec{p}\rangle_c \equiv \sqrt{2E_p}|\vec{p}\rangle$. Therefore our formulas for matrix elements $\langle\vec{p}|\dots|\vec{p}\rangle$ have a factor of $2E_p$, which does not appear in covariant normalization. Note that in our non-covariant normalization, the matrix elements J^μ_{FI} are dimensionless.

(1) Elastic scattering ($x = 1$):

What is the form of the transition current J^μ_{FI} ? If the proton were a point particle, it must be of the same form as for the electron, i.e., $J^\mu_{FI} = \sqrt{\frac{M}{E_{P'}}}\sqrt{\frac{M}{E_P}}\,\overline{u}(\vec{P}', S')\,\gamma^\mu\, u(\vec{P}, S)$. The effects of proton structure can be described by using a modified vertex:

$$\gamma^\mu \longrightarrow \Gamma^\mu(P', P) \tag{10.18}$$

This vertex must contain the electric and magnetic form factors, and will be specified later. So, we write for the proton transition current

$$J^\mu_{FI}(P', P) = \sqrt{\frac{M}{E_{P'}}}\sqrt{\frac{M}{E_P}}\,\overline{u}(\vec{P}', S')\,\Gamma^\mu(P', P)\,u(\vec{P}, S) \tag{10.19}$$

Consider now the hadronic tensor (10.16):

$$\begin{aligned} W^{\mu\nu} &= 2M^2 \int \frac{\mathrm{d}^3 P'}{2E_{P'}}\,\delta^{(4)}\left(P' + p' - P - p\right) \sum_{S'} \left(\overline{u}(\vec{P}', S')\,\Gamma^\mu(P', P)\,u(\vec{P}, S)\right) \\ &\times \left(\overline{u}(\vec{P}', S')\,\Gamma^\nu(P', P)\,u(\vec{P}, S)\right)^* \end{aligned} \tag{10.20}$$

We can rewrite this as a 4-dimensional momentum integral by using the identity [2]

$$\int \frac{\mathrm{d}^3 P'}{2E_{P'}} = \int \mathrm{d}^4 P'\,\delta\left(P'^2 - M^2\right)\,\theta(P'_0)$$

Then the hadronic tensor (10.25) becomes

$$\begin{aligned} W^{\mu\nu} &= 2M^2\,\theta(P_0 + p_0 - p'_0)\,\delta\left[(P + p - p')^2 - M^2\right] \\ &\times \sum_{S'} \left(\overline{u}(\vec{P}', S')\,\Gamma^\mu(P', P)\,u(\vec{P}, S)\right)\,\left(\overline{u}(\vec{P}', S')\,\Gamma^\nu(P', P)\,u(\vec{P}, S)\right)^* \end{aligned}$$

where $P' = P + p - p'$.

In order to calculate the usual differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\,\Omega'} = \int_m^\infty \frac{\mathrm{d}\sigma}{\mathrm{d}E_{p'}\,\mathrm{d}\Omega'}\,\mathrm{d}E_{p'}$$

from Eq.(10.17), we also have to integrate over the final electron energy $E_{p'}$. This can be done by using the following relation (in the laboratory frame):

$$\int_m^\infty \mathrm{d}E_{p'}\,\theta(M + E_p - E_{p'})\,\delta\left[(P + p - p')^2 - M^2\right]\,F(E_{p'}) = \frac{1}{2M}\,\frac{1}{1 + \frac{2E_p}{M}\sin^2\frac{\theta}{2}}\,F(E_{p'} = E') \tag{10.21}$$

[2]This is an identity, because on the r.h.s. the integration over P'_0 gives: $\int_{-\infty}^\infty \mathrm{d}P'_0\,\delta\left(P'^2 - M^2\right)\theta(P'_0) = \frac{1}{2E_{P'}}\int_{-\infty}^\infty \mathrm{d}P'_0\delta(P'_0 - E_{P'}) = \frac{1}{2E_{P'}}$.

where $F(E_{p'})$ is any function of $E_{p'}$, and the final electron energy E' is given by the initial electron energy ($E = E_p$) and the scattering angle by (see also Sect. 9, Eq.(9.8)):

$$E' = \frac{E}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}} \tag{10.22}$$

<u>Home work</u>: Derive Eq.(10.21) by using the following identity in the laboratory frame:

$$(P + p - p')^2 - M^2 = 2M(E_p - E_{p'}) - 4E_pE_{p'}\sin^2\frac{\theta}{2}$$

(Remember that we always neglect the electron mass: $E_p \simeq |\vec{p}|$ and $E_{p'} \simeq |\vec{p}\,'|$.)

Then we get the following relation for the differential <u>unpolarized</u> cross section (with $\alpha = e^2/(4\pi)$) [3]:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'} = \frac{\alpha^2}{4}\frac{E'}{E}\frac{1}{M^2\,q^4}\frac{1}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}}\sum_{sS}\ell_{\mu\nu}\,w^{\mu\nu} \tag{10.23}$$

where the leptonic and hadronic tensors (for the elastic case) are given by

$$\ell^{\mu\nu} = 2m^2\sum_{s'}\left(\overline{u}(\vec{p}\,', s')\gamma^\mu u(\vec{p}, s)\right)\left(\overline{u}(\vec{p}\,', s')\gamma^\nu u(\vec{p}, s)\right)^* \tag{10.24}$$

$$w^{\mu\nu} = 2M^2\sum_{S'}\left(\overline{u}(\vec{P}', S')\Gamma^\mu u(\vec{P}, S)\right)\left(\overline{u}(\vec{P}', S')\Gamma^\nu u(\vec{P}, S)\right)^* \tag{10.25}$$

What is the form of the <u>effective vertex</u> Γ^μ? Remember: For a point particle (electron) we have $\Gamma^\mu = \gamma^\mu$. What is the difference between electron and proton (besides opposite charge)?

- Spin g-factor (magnetic moment): For electron $g_s = 2$, but for proton $g_s = 5.58$.
- Proton has an extended charge distribution. Its Fourier transform is the electric form factor [4] $G_E(Q^2)$.
- Proton has an extended magnetic moment distribution. Its Fourier transform is the magnetic form factor $G_M(Q^2)$.

Note: In Sect. 6 we used the notations F_e and F_m for the electric and magnetic form factors, but from now we use G_E and G_M.

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[3] The cross section for <u>unpolarized</u> scattering is obtained from (10.17) by an average over initial spins and sum over final spins: $\frac{1}{4}\sum_{sS}$

[4] The form factors must be Lorentz invariant, and therefore depend on the 4-momentum transfer squared: $G_E(Q^2)$ and $G_M(Q^2)$, where $Q^2 = -q^2 = \vec{q}^{\,2} - q_0^2 > 0$.

To get some hint about the form of the effective vertex Γ^μ, we first proof the following identity for the electron transition current (Gordon identity):

$$j^\mu_{fi} = \sqrt{\frac{m}{E_k}}\sqrt{\frac{m}{E_{k'}}}\,\overline{u}(\vec{k}',s')\gamma^\mu\,u(\vec{k},s) = \sqrt{\frac{m}{E_k}}\sqrt{\frac{m}{E_{k'}}}\,\overline{u}(\vec{k}',s')\left[\frac{(k'+k)^\mu}{2M} + i\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]u(\vec{k},s) \tag{10.26}$$

Here $q = k' - k$, and the Dirac matrix $\sigma^{\mu\nu}$ is defined by the following commutator between γ-matrices:

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \tag{10.27}$$

Proof of (10.26): Use the following identity for any 4-vectors a^μ and b^μ:

$$\begin{aligned} \not{a}\not{b} &= a_\mu\, b_\nu \frac{1}{2}\left[(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right] \\ &= a_\mu b^\mu - i\sigma^{\mu\nu}\,a_\mu b_\nu \end{aligned} \tag{10.28}$$

By using the Dirac equation $(\not{k} - m)\,u(\vec{k},s) = \overline{u}(\vec{k}',s')\,(\not{k}' - m) = 0$, we then have for any 4-vector a^μ

$$\begin{aligned} 0 &= \overline{u}(\vec{k}',s')\left[(\not{k}' - m)\not{a} + \not{a}\,(\not{k} - m)\right]u(\vec{k},s) \\ &= \overline{u}(\vec{k}',s')\left[k'\cdot a - i\sigma^{\mu\nu}k'_\mu a_\nu + k\cdot a - i\sigma^{\mu\nu}a_\mu k_\nu - 2m\not{a}\right]u(\vec{k},s) \\ &= 2m\,\overline{u}(\vec{k}',s')\left[\frac{(k'+k)^\mu}{2m}a_\mu + i\frac{\sigma^{\mu\nu}a_\mu q_\nu}{2m} - \gamma^\mu a_\mu\right]u(\vec{k},s) \end{aligned}$$

Because a^μ is arbitrary, the Gordon formula (10.26) follows.

What is the physical meaning of the 2 terms on the r.h.s. of (10.26)? Consider first the nonrelativistic limit, where the Dirac spinor is given by $u(\vec{k},s) = \left(\chi(s), \frac{\vec{\sigma}\cdot\vec{k}}{2m}\chi(s)\right)$ with $\chi(s)$ the 2-component Pauli spinor. Then we get from (10.26) in the nonrelativistic limit (lowest order in $1/m$)

$$\overline{u}(\vec{k}',s')\gamma^0 u(\vec{k},s) \simeq \chi^\dagger(s')\chi(s) = \delta_{s's} \tag{10.29}$$

$$\overline{u}(\vec{k}',s')\vec{\gamma}u(\vec{k},s) \simeq \chi^\dagger(s')\left[\frac{(\vec{k}+\vec{k}')}{2m} + i\frac{\vec{\sigma}\times\vec{q}}{2m}\right]\chi(s) \tag{10.30}$$

Here we used the relation $\sigma^{ij} = \epsilon^{ijk}\Sigma^k$, where $\vec{\Sigma}$ is the relativistic spin matrix $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$. We see that (10.29) is the (transition) charge density for a point particle in momentum space, the first term of Eq.(10.30) is the convection current density, and the second term of (10.30) is the spin current density in momentum space for a point particle with $g_s = 2$: The latter can be written in the form $\vec{j}_{\rm spin} = i\,(\vec{\mu}\times\vec{q})$ where the magnetic moment is given by (see Sect. 11 of spring semester, Eq.(11.10)): $\vec{\mu} = \frac{1}{2m}\left(\chi^\dagger(s')\vec{\sigma}\chi(s)\right)\frac{g_s}{2}$ with $g_s = 2$.

Returning to the relativistic expression (10.26), we can roughly say that the first term on the r.h.s. describes the convection current, and the second term describes the spin current. For a point particle, these two terms must go just in the combination of Eq.(10.26), but for the proton (which has a size, and a different spin g-factor), these two terms will go in some other combination, involving the electric and magnetic form factors. To get this combination for the proton, let us fist multiply the two terms on the r.h.s. of (10.26) by independent form factors, which we call [5] $F_1(Q^2)$ and $(F_1(Q^2)+F_2(Q^2))$:

$$\begin{aligned} J^\mu_{FI} &= \sqrt{\frac{M}{E_P}}\sqrt{\frac{M}{E_{P'}}}\,\overline{u}(\vec{P}',S')\,\Gamma^\mu(P',P)\,u(\vec{P},S) \\ &= \sqrt{\frac{M}{E_P}}\sqrt{\frac{M}{E_{P'}}}\,\overline{u}(\vec{P}',S')\left[\frac{(P'+P)^\mu}{2M}\,F_1(Q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2M}\,\left(F_1(Q^2)+F_2(Q^2)\right)\right]u(\vec{P},S) \end{aligned} \tag{10.31}$$

where $Q^2 = -q^2 = \vec{q}^{\,2} - q_0^2$. How are these form factors F_1 and F_2 related to the electric and magnetic form factors defined in Sects. 7 and 8 for a static proton? From the previous Eqs.(7.5) and (8.5), we see that the electric and magnetic form factors were defined as ordinary 3-dimensional Fourier transforms of the charge and magnetic moment densities, which is possible if the form factors are functions of only the 3-momentum transfer ($\vec{q}^{\,2}$), not the 4-momentum transfer ($Q^2 = \vec{q}^{\,2} - q_0^2$). Therefore, just for the present purpose of physical interpretation, we consider the transition current (10.31) in a special Lorentz frame where $q_0 = 0$. This frame is called the <u>Breit frame</u>, which is defined as follows: The incoming momentum is $\vec{P} = -\vec{q}/2$, and the outgoing momentum is $\vec{P}' = \vec{P}+\vec{q} = \vec{q}/2$. Because in this frame $\vec{P}+\vec{P}' = 0$, and $E_P = E_{P'} = E_{q/2}$, the current (10.31) takes the following very simple form:

$$J^0_{FI}(\text{Breit}) = \frac{M}{E_{q/2}}\left(F_1(Q^2) - \frac{Q^2}{4M^2}\,F_2(Q^2)\right)\delta_{S',S} \tag{10.32}$$

$$\vec{J}_{FI}(\text{Breit}) = \frac{M}{E_{q/2}}\left[\chi^\dagger(S')\,i\,\frac{(\vec{\sigma}\times\vec{q})}{2M}\chi(S)\right]\left(F_1(Q^2)+F_2(Q^2)\right) \tag{10.33}$$

where $Q^2 = \vec{q}^{\,2}$ in the Breit frame.

<u>Home work:</u> Use the explicit form of the Dirac spinors, given in Sect. 4 (Eq.(4.16)), in the Breit

[5] For the electron (Dirac point-particle) we have $F_1 = 1$ and $F_2 = 0$.

frame to derive (10.32) and (10.33) from (10.31).

Besides the relativistic correction factor $\frac{M}{E_{q/2}}$, we see that in this frame (i) the time component (10.32) is equal to the nonrelativistiv form (10.29), multiplied by the form factor combination $F_1 - (Q^2/4M^2)F_2$, and (ii) the space component (10.33) is equal to the nonrelativistic form of the spin current [6] in (10.30), multiplied by the form factor combination $(F_1 + F_2)$. Therefore we can identify the electric and magnetic form factors as

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2) \tag{10.34}$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2) \tag{10.35}$$

To get the correct charge and spin g - factor of the proton, the normalizations are $G_E(0) = 1$, $G_M(0) = \frac{g_s^{(p)}}{2}$ with $g_s^{(p)} = 5.58$. This corresponds to $F_1(0) = 1$, $F_2(0) = \frac{g_s^{(p)}}{2} - 1 \equiv \kappa^{(p)}$, where $\kappa^{(p)} = 1.79$ is called the "anomalous part" of $\frac{g_s^{(p)}}{2}$. For the neutron we have $G_E(0) = 0$, $F_M(0) = \frac{g_s^{(n)}}{2}$ with $g_s^{(n)} = -3.82$. This corresponds to $F_1(0) = 0$, $F_2(0) = \frac{g_s^{(n)}}{2} - 1 \equiv \kappa^{(n)}$, where $\kappa^{(n)} = -1.91$.

<u>Summary of results for the proton (or neutron) transition current:</u>

- The transition current is given by (10.31), where the form factors F_1 and F_2 are related to the electric and magnetic form factors G_E and G_M by (10.34) and (10.35).
- By using the Gordon formula (10.26), we can rewrite the spinor matrix elements in (10.31) in the following equivalent forms:

$$\overline{u}\,\Gamma^\mu\, u = \overline{u}(\vec{P}', S') \left[\gamma^\mu F_1(Q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2M} F_2(Q^2) \right] u(\vec{P}, S) \tag{10.36}$$

$$= \overline{u}(\vec{P}', S') \left[\gamma^\mu \left(F_1(Q^2) + F_2(Q^2) \right) - \frac{(P + P')^\mu}{2M} F_2(Q^2) \right] u(\vec{P}, S) \tag{10.37}$$

Because of the form given in Eq.(10.36), $F_1(Q^2)$ is called the "Dirac form factor", and $F_2(Q^2)$ is called the "Pauli form factor". The form of Eq.(10.37) is most convenient for calculations.

<u>Calculation of elastic cross section</u>:

We go back to (10.23) to calculate the relativistic elastic cross section. Using the leptonic tensor

[6] Note that the convection current is zero in the Breit frame, because $\vec{p} + \vec{p}' = 0$.

(10.24), and the method to perform spin sums by traces of Dirac matrices (see Sect. 6), we get

$$\begin{aligned}\sum_s \ell^{\mu\nu} &= 2m^2 \sum_{ss'} \left(\overline{u}(\vec{p}', s')\gamma^\mu u(\vec{p}, s)\right) \left(\overline{u}(\vec{p}', s')\gamma^\nu u(\vec{p}, s)\right)^* \\ &= \frac{1}{2} \mathrm{Tr}\left[\left(\not{p}' + m\right) \gamma^\mu \left(\not{p} + m\right) \gamma^\nu\right]\end{aligned}$$

Using the two theorems about traces of Dirac matrices (see Sect. 6, Eqs.(6.22) - (6.24)), we obtain

$$\sum_s \ell^{\mu\nu} = 2\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}\left(p' \cdot p - m^2\right)\right] \tag{10.38}$$

Using the hadronic tensor (10.25), we get

$$\begin{aligned}\sum_S w^{\mu\nu} &= 2M^2 \sum_{SS'} \left(\overline{u}(\vec{P}', S')\Gamma^\mu u(\vec{P}, S)\right) \left(\overline{u}(\vec{P}', S')\Gamma^\nu u(\vec{P}, S)\right)^* \\ &= \frac{1}{2} \mathrm{Tr}\left[\left(\not{P}' + M\right) \Gamma^\mu \left(\not{P} + M\right) \Gamma^\nu\right] \\ &\equiv w_1^{\mu\nu} + w_2^{\mu\nu}\end{aligned} \tag{10.39}$$

Here we split Γ^μ into two terms as given in Eq.(10.37). The part $w_1^{\mu\nu}$ is similar to the electron case and given by

$$w_1^{\mu\nu} = 2\left(F_1(Q^2) + F_2(Q^2)\right)^2 \left[P'^\mu P^\nu + P'^\nu P^\mu - g^{\mu\nu}\left(P' \cdot P - M^2\right)\right] \tag{10.40}$$

The part $w_2^{\mu\nu}$ contains all the other pieces, and is given by

$$\begin{aligned}w_2^{\mu\nu} &= -\frac{1}{4M} F_2(Q^2)\left(F_1(Q^2) + F_2(Q^2)\right) \\ &\times \left[(P' + P)^\mu \,\mathrm{Tr}\left[\left(\not{P}' + M\right)\left(\not{P} + M\right)\gamma^\nu\right] + (P' + P)^\nu \,\mathrm{Tr}\left[\left(\not{P}' + M\right)\gamma^\mu\left(\not{P} + M\right)\right]\right] \\ &+ \frac{1}{8M^2}\left(F_2(Q^2)\right)^2 (P' + P)^\mu (P' + P)^\nu \,\mathrm{Tr}\left[\left(\not{P}' + M\right)\left(\not{P} + M\right)\right] \\ &= (P' + P)^\mu (P' + P)^\nu \left[-2F_2(Q^2)\left(F_1(Q^2) + F_2(Q^2)\right) + (F_2(Q^2))^2 \frac{P' \cdot P + M^2}{2M^2}\right]\end{aligned} \tag{10.41}$$

Finally, one has to calculate the contraction $\sum_{sS} \ell_{\mu\nu} w^{\mu\nu}$ by using (10.38) for the electron part, and the sum of (10.40) and (10.41) for the proton part. Inserting the result into (10.23) gives the Rosenbluth formula of Sect. 8, Eq.(8.11):

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} \left[\frac{\left(G_E(Q^2)\right)^2 + b\left(G_M(Q^2)\right)^2}{1 + b} + 2b\left(G_M(Q^2)\right)^2 \tan^2\frac{\theta}{2}\right] \tag{10.42}$$

where $Q^2 = -q^2 = \vec{q}^2 - q_0^2 > 0$ is the 4-momentum transfer squared, and $b = Q^2/(4M^2)$.
Home work (a bit long . . .): Calculate the contraction $\sum_{sS} \ell_{\mu\nu} w^{\mu\nu}$, and insert it into (10.23) to derive (10.42).

The comparison of Eq.(10.42) with experimental data gives the experimentally measured electric and magnetic form factors, as has been discussed already in Sect. 8.