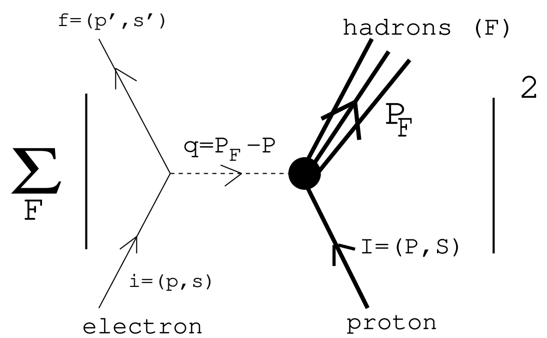
11 Inelastic inclusive electron-proton scattering

Here we consider inelastic inclusive scattering, where the Bjorken variable $x = -q^2/(2P \cdot q) < 1$, and the final hadronic state is not observed in the experiment $\Rightarrow$ This means a sum over all possible final hadronic states [1] ($F = N + \pi,\ N + 2\pi, \Sigma + K, \dots$).



11.1 Cross section and hadronic tensor

The differential cross section for this case is obtained from Sect. 10, Eq.(10.17):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_{p'}\,\mathrm{d}\Omega'} = \frac{e^4}{8\pi^2}\frac{p'}{p}\frac{1}{E_P\,q^4}\,\ell_{\mu\nu}W^{\mu\nu} \tag{11.1}$$

Here the leptonic and hadronic tensors are defined by (see Eqs.(10.15) and (10.16)):

$$\ell^{\mu\nu} = 2m^2\sum_{s'}\left(\overline{u}(\vec{p}',s')\gamma^\mu u(\vec{p},s)\right)\left(\overline{u}(\vec{p}',s')\gamma^\nu u(\vec{p},s)\right)^* \tag{11.2}$$

$$W^{\mu\nu} = E_P\sum_F\int \mathrm{d}^3P_F\,\delta^{(4)}\left(P_F - P - q\right)\,J^\mu_{FI}\,J^{\nu *}_{FI} \tag{11.3}$$

The difference to the previous elastic case (Eq.(10.16)) is that, in the Hadronic tensor (11.3), we now have to sum over all possible hadronic states (F). In order to perform this sum, we first note that

$$(2\pi)^4\ \delta^{(4)}(P_F - P - q) = \int \mathrm{d}^4z\,e^{-i(P_F-P-q)\cdot z}$$

and re-introduce the x - dependent hadron transition current as (see Sect. 10, Eq.(10.8))

$$J^\mu_{FI}(x) = e^{i(P_F-P)\cdot x}\,J^\mu_{FI}(x=0) \tag{11.4}$$

[1] The single proton state (elastic scattering, $x = 1$) is automatically excluded by the restriction to $x < 1$.

Then we can express the hadronic tensor (11.3) as follows:

$$\begin{aligned}
W^{\mu\nu} &= \frac{E_P}{2\pi}\sum_F \int \frac{\mathrm{d}^3 P_F}{(2\pi)^3} \int \mathrm{d}^4 z \, e^{-i(P_F-P)\cdot z} \, e^{iq\cdot z} \; J^{\mu}_{FI} \, J^{\nu *}_{FI} \\
&= \frac{E_P}{2\pi}\sum_F \int \frac{\mathrm{d}^3 P_F}{(2\pi)^3} \int \mathrm{d}^4 z \, e^{iq\cdot z} \, J^{\mu}_{FI} \, J^{\nu *}_{FI}(z) \\
&\equiv \frac{E_P}{2\pi}\hat{\sum}_F \int \mathrm{d}^4 z \, e^{iq\cdot z} \, J^{\mu}_{FI} \, J^{\nu *}_{FI}(z) \qquad (11.5)
\end{aligned}$$

where the sum $\hat{\sum}_F$ includes the integration over the total 3-momentum of the hadrons:

$$\hat{\sum}_F \equiv \sum_F \int \frac{\mathrm{d}^3 P_F}{(2\pi)^3}$$

The transition current J^{μ}_{FI}, by definition, is the transition matrix element of the current operator ($\hat{J}^{\mu}$) between the hadronic states F and I:

$$J^{\mu}_{FI} = \langle F|\hat{J}^{\mu}|I\rangle \, , \qquad J^{\nu *}_{FI}(z) = \langle F|\hat{J}^{\nu}(z)|I\rangle^* = \langle I|\hat{J}^{\nu}(z)|F\rangle \qquad (11.6)$$

[In the last step, we used the hermiticity of the current operator $\hat{J}^{\nu}$: For any operator $\hat{O}$ we have the identity $\langle f|\hat{O}|i\rangle^* = \langle f|\hat{O}^{\dagger}|i\rangle$, and if $\hat{O}$ is hermite then $\hat{O}^{\dagger} = \hat{O}$.]

Then the hadronic tensor (11.5) becomes

$$\begin{aligned}
W^{\mu\nu} &= \frac{E_P}{2\pi} \int \mathrm{d}^4 z \, e^{iq\cdot z} \hat{\sum}_F \langle I|\hat{J}^{\nu}(z)|F\rangle \, \langle F|\hat{J}^{\mu}|I\rangle \\
&= \frac{E_P}{2\pi} \int \mathrm{d}^4 z \, e^{iq\cdot z} \langle I|\hat{J}^{\nu}(z) \, \hat{J}^{\mu}(0)|I\rangle \qquad (11.7)
\end{aligned}$$

where we used the completeness of hadronic states:

$$\hat{\sum}_F |F\rangle \, \langle F| = 1 \qquad (11.8)$$

We then get the important result: The hadronic tensor for inelastic inclusive scattering is the Fourier transform of the "current-current correlation function" in the initial proton state $|I\rangle = |P, S\rangle$:

$$W^{\mu\nu} = \frac{2E_P}{4\pi} \int \mathrm{d}^4 z \, e^{iq\cdot z} \langle P, S|\hat{J}^{\nu}(z) \, \hat{J}^{\mu}(0)|P, S\rangle \qquad (11.9)$$

As explained in Sect. 10, the factor $2E_P$ in this formula reflects our non-covariant normalization of state vectors, and is absent in covariant normalization.

11.2 Structure functions

Change of notation: From now, we denote the 4-momentum of the initial proton by p^μ and its spin 4-vector by S^μ. The 4-momentum of the initial electron is denoted by k^μ and its spin 4-vector by s^μ. The 4-momentum transfer is $q = k - k'$. The initial and final electron energies are denoted by E and E'.

The hadronic tensor (11.9) depends on the 4-vectors p^μ, q^μ, and S^μ. Current conservation $\partial_\nu \hat{J}^\nu(z) = 0$ leads to the following condition on the hadonic tensor: If we multiply (11.9) by q_ν (summation over ν) and perform a partial integration in z, we obtain the condition [2]

$$q_\nu W^{\mu\nu} = q_\mu W^{\mu\nu} = 0 \tag{11.10}$$

The most general form of the Lorentz tensor $W^{\mu\nu}$, consistent with (11.10) and also time-reversal invariance, is then as follows:

$$\begin{aligned} W^{\mu\nu} &= W_1(x,Q^2)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2(x,Q^2)}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right) && (11.11)\\ &+ M\, i\epsilon^{\mu\nu\lambda\sigma}\, q_\lambda \left[G_1(x,Q^2)\, S_\sigma + \frac{G_2(x,Q^2)}{M^2}\,(p\cdot q\, S_\sigma - (S\cdot q)\, p_\sigma)\right] && (11.12)\end{aligned}$$

Here W_1, W_2, G_1, G_2 are functions of the Lorentz invariant variables $Q^2 = -q^2 > 0$ and $x = Q^2/(2p\cdot q)$. The are called structure functions. $\epsilon^{\mu\nu\lambda\sigma}$ is totally antisymmetric in all 4 Lorentz indices, with the definition $\epsilon^{0123} = 1$. We see that $W^{\mu\nu}$ has a symmetric part which is independent of the proton spin direction, and an antisymmetric part which depends on the spin direction:

$$W^{\mu\nu} = W^{\mu\nu}_{(S)} + W^{\mu\nu}_{(A)} \tag{11.13}$$

Remember (spring semester, Sect. 7, Eq.(7.7)): The spin 4-vector of the proton has the form

$$S^\mu = \left(\frac{\vec{p}\cdot\vec{S}}{M}, \vec{S} + \frac{\vec{p}}{M}\frac{\vec{p}\cdot\vec{S}}{E_p + M}\right) \tag{11.14}$$

where the unit vector [3] $\vec{S}$ is the spin direction in the rest frame, which is considered as a generic (fixed) direction in space. The spin 4-vector satisfies the relations $S^2 = -1$, $p \cdot S = 0$.

[2] The second equality in (11.10) can be derived by following the above arguments with slight modifications (shifting the z-dependence to the current $\hat{J}^\mu$ instead of $\hat{J}^\nu$).

[3] Here we use the normalization $|\vec{S}| = |\vec{s}| = 1$ for the spin vectors of the proton and the electron in the rest fram, while in Sect. 7 we normalized them by 1/2. Also, in Sect. 7, we denoted the spin direction in the rest frame by $\vec{S}_0$, while here we simply write $\vec{S}$.

For reasons which will become clear later, it is sometimes convenient to use another set of dimensionless structure functions F_1, F_2, g_1, g_2, which are related to the above ones by

$$
\begin{aligned}
F_1(x,Q^2) &= W_1(x,Q^2)\,, \qquad F_2(x,Q^2) = \frac{p\cdot q}{M^2}\, W_2(x,Q^2)\,,\\
g_1(x,Q^2) &= (p\cdot q)\, G_1(x,Q^2)\,, \qquad g_2(x,Q^2) = \frac{(p\cdot q)^2}{M^2}\, G_2(x,Q^2)
\end{aligned}
$$

The hadronic tensor is then expressed by these new structure functions as

$$
\begin{aligned}
W^{\mu\nu} &= F_1(x,Q^2)\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+\frac{F_2(x,Q^2)}{p\cdot q}\left(p^\mu-\frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu-\frac{p\cdot q}{q^2}q^\nu\right) && (11.15)\\
&+ \frac{M}{p\cdot q}\, i\epsilon^{\mu\nu\lambda\sigma}\, q_\lambda\left[g_1(x,Q^2)\,S_\sigma+g_2(x,Q^2)\left(S_\sigma-\frac{(S\cdot q)\,p_\sigma}{p\cdot q}\right)\right] && (11.16)
\end{aligned}
$$

We can use the definition of Bjorken x in these relations to replace $p\cdot q = \frac{Q^2}{2x}$.

11.3 Leptonic tensor

In order to calculate the cross section (11.1) including the case of polarized electron and proton in the initial state, we also need the form of the leptonic tensor (11.2) [4]. In order to reduce this to a calculation of Dirac traces, we use the spin projection operator for the electron, see Sect. 9 of spring semester, p.4 [5]:

$$
P(s) = \frac{1}{2}\left(1+\gamma_5 \not{s}\right) \tag{11.17}
$$

It satisfies the relation (see Sect. 9)

$$
P(s)u(\vec{k},s') = \delta_{ss'}\, u(\vec{k},s')
$$

Then the leptonic tensor (11.2), for fixed initial electron spin component $s=\pm 1/2$, can be expressed as

$$
\begin{aligned}
\ell^{\mu\nu} &= 2m^2\sum_{s's_0}\left(\overline{u}(\vec{k}',s')\gamma^\mu P(s)u(\vec{k},s_0)\right)\left(\overline{u}(\vec{k}',s')\gamma^\nu P(s)u(\vec{k},s_0)\right)^*\\
&= \frac{1}{2}\mathrm{Tr}\left[(\not{k}'+m)\,\gamma^\mu P(s)\,(\not{k}+m)\,P(s)\gamma^\nu\right]
\end{aligned}
$$

[4] In Sect. 10 (p. 9) we calculated the leptonic tensor for the case of unpolarized scattering, i.e., including the summation over s. If the initial electron is polarized, we should not sum over s.

[5] Remember that here we normalize the spin 4-vector to $s^2=-1$, while in Sect. 9 we used $s^2=-1/4$.

Here we used the methods of Sect. 6 to convert spin sums to traces over Dirac matrices. If we use the relation $\not{s}\not{k} = -\not{k}\not{s}$ (which follows from $k \cdot s = 0$), we see that $\not{k}P(s) = P(s)\not{k}$. Using also $P^2(s) \equiv P(s)$ (because it is a projection operator), we see that

$$P(s)\,(\not{k} + m)\,P(s) = P(s)\,(\not{k} + m) = (\not{k} + m)\,P(s)$$

Then we obtain

$$\ell^{\mu\nu} = \frac{1}{4}\mathrm{Tr}\left[(\not{k}' + m)\,\gamma^\mu\,(1 + \gamma_5\not{s})\,(\not{k} + m)\,\gamma^\nu\right] \tag{11.18}$$

Now we can calculate the trace (Tr) over the Dirac matrices, using the theorems given in Sect. 6, Eqs. (6.22), (6.23), (6.24), plus two additional theorems about the matrix $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. For convenience, we list all 5 theorems here:

$$\mathrm{Tr}\,(\Gamma_1 \ldots \Gamma_n) = 0 \qquad \text{if } n = \text{odd} \tag{11.19}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4\,g^{\mu\nu} \tag{11.20}$$

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right) \tag{11.21}$$

$$\mathrm{Tr}\,(\gamma_5\gamma^\mu\gamma^\nu) = 0 \tag{11.22}$$

$$\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\right) = -4i\epsilon^{\mu\nu\sigma\lambda} \tag{11.23}$$

The result for the trace (11.18) is as follows:

$$\ell^{\mu\nu} = k'^\mu k^\nu + k'^\nu k^\mu - g^{\mu\nu}\left(k' \cdot k - m^2\right) - im\epsilon^{\mu\nu\sigma\lambda}q_\sigma s_\lambda \tag{11.24}$$

<u>Home work</u>: Derive the result (11.24) from (11.18), by using the theorems given in (11.19 - (11.23). In the following, we will take the limit $m \to 0$. Then (see Eq.(11.14))

$$m\,s^\mu \xrightarrow{m\to 0} \left(\vec{k}\cdot\vec{s},\, \vec{k}(\hat{\vec{k}}\cdot\vec{s})\right) = (\hat{\vec{k}}\cdot\vec{s})\left(E, \vec{k}\right) = (\hat{\vec{k}}\cdot\vec{s})k^\mu \tag{11.25}$$

Most experiments are done for <u>longitudinal electron polarization</u>: $\hat{\vec{k}}\cdot\vec{s} = \pm 1$, which is called positive (or negative) helicity. In this case,

$$ms^\mu = \pm k^\mu \tag{11.26}$$

We then split the leptonic tensor (11.24) into a symmetric part ($\ell_S^{\mu\nu}$) and an antisymmetric part ($\ell_A^{\mu\nu}$):

$$\ell^{\mu\nu} = \ell_{(S)}^{\mu\nu} \pm \ell_{(A)}^{\mu\nu} \tag{11.27}$$

where the upper (lower) sign refers to positive (negative) electron helicity, and (using $q = k - k'$)

$$\ell^{\mu\nu}_{(S)} = k'^{\mu} k^{\nu} + k'^{\nu} k^{\mu} - g^{\mu\nu} (k' \cdot k) \qquad (11.28)$$

$$\ell^{\mu\nu}_{(A)} = i\epsilon^{\mu\nu\sigma\lambda} k'_{\sigma} k_{\lambda} \qquad (11.29)$$

11.4 Calculation of cross section

By using the expressions for the hadronic tensor, Eq.(11.11), (11.12), and the leptonic tensor, Eq.(11.28), (11.29), we can now calculate the cross section (11.1). In the calculation, we can make use of

$$\ell^{\mu\nu} W_{\mu\nu} = \ell^{\mu\nu}_{(S)} W_{(S)\mu\nu} \pm \ell^{\mu\nu}_{(A)} W_{(A)\mu\nu} \qquad (11.30)$$

Therefore the cross section (11.1) splits into a spin independent part ("unpolarized part") and a spin dependent part ("polarized part"):

$$\frac{\mathrm{d}\sigma^{\pm}}{\mathrm{d}E'\,\mathrm{d}\Omega'} = \frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}E'\,\mathrm{d}\Omega'} \pm \frac{\mathrm{d}\sigma^{A}}{\mathrm{d}E'\,\mathrm{d}\Omega'} \qquad (11.31)$$

Here $\pm$ refers to the helicity of the incoming electron. If the incoming electron is unpolarized, we must average over the helicities, and the cross section becomes simply $\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}E'\,\mathrm{d}\Omega'}$.

(1) Unpolarized cross section

Here we calculate the contraction $\ell^{\mu\nu}_{(S)} W_{(S)\mu\nu}$ of Eq.(11.30). Because of $q^{\mu} W_{(S)\mu\nu} = 0$ (where $q = k - k'$), we can replace $k' \to k$ in the leptonic tensor (11.28) [6] :

$$\ell^{\mu\nu}_{(S)} W_{(S)\mu\nu} = \left(2k^{\mu} k^{\nu} + \frac{q^2}{2} g^{\mu\nu} \right) W_{(S)\mu\nu}$$

For the two terms in $W_{(S)\mu\nu}$ of Eq.(11.11), we get the following results:

$$\left(2k^{\mu} k^{\nu} + \frac{q^2}{2} g^{\mu\nu} \right) \left(-g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{q^2} \right) = -q^2 \qquad (11.32)$$

$$\left(2k^{\mu} k^{\nu} + \frac{q^2}{2} g^{\mu\nu} \right) \left(p_{\mu} - \frac{p \cdot q}{q^2} q_{\mu} \right) \left(p_{\nu} - \frac{p \cdot q}{q^2} q_{\nu} \right) = 2(p \cdot k)(p \cdot k') + \frac{q^2}{2} M^2 \qquad (11.33)$$

Homework: Verify Eq.(11.32) and (11.33), by using the on mass-shell relations $k'^2 = k^2 = m^2 \simeq 0 \Rightarrow k \cdot q = -q^2/2$.

[6] As always, we use $k'^2 = (k+q)^2 = k^2 + q^2 + 2k \cdot q$ together with the on-shell conditions $k'^2 = k^2 = m^2$.

In the laboratory frame where $p^\mu = (M, \vec{0})$, (11.33) simplifies:

$$2(p \cdot k)(p \cdot k') + \frac{q^2}{2} M^2 = 2EE'\, M^2 \cos^2 \frac{\theta}{2}$$

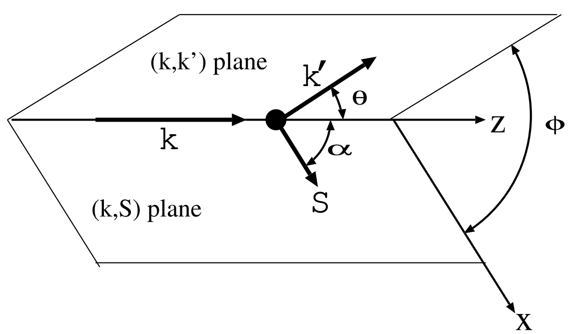
where we used $q^2 = -4EE' \sin^2 \frac{\theta}{2}$ (see Sect.9, Eq.(9.7)). We then finally obtain the unpolarized cross section as

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}E'\,\mathrm{d}\Omega'} = \alpha^2 \frac{4E'^2}{MQ^4} \left(2W_1(x, Q^2) \sin^2 \frac{\theta}{2} + W_2(x, Q^2) \cos^2 \frac{\theta}{2} \right) \tag{11.34}$$

Here $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ is the fine structure constant.

(2) Polarized cross section

In this case we have not only the angle (θ) between $\vec{k}$ and $\vec{k}'$, but also the angle (α) between $\vec{k}$ and $\vec{S}$, and the angle (ϕ) between $\vec{k}'$ and $\vec{S}$. We therefore chose the following coordinate system (in the laboratory frame):



The vector $\vec{k}$ is in the z-direction, and the vector $\vec{S}$ is in the (x, z) plane. (θ, ϕ) are the polar and azimuthal angles of $\vec{k}'$. Then the 4-vectors k^μ, k'^μ, and S^μ take the following form:

$$\begin{aligned} k^\mu &= E\,(1, 0, 0, 1) \\ k'^\mu &= E'\,(1, \sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta) \\ S^\mu &= (0,\, \sin\alpha,\, 0,\, \cos\alpha) \end{aligned}$$

By using these 4-vectors, the contraction

$$\ell^{\mu\nu}_{(A)} W_{(A)\mu\nu} = \left((i) \epsilon^{\mu\nu\sigma\lambda} k'_\sigma k_\lambda \right) W_{(A)\mu\nu}$$

can be calculated. For the two terms in (11.12), we obtain the following results:

$$\begin{aligned}\left(\epsilon^{\mu\nu\sigma\lambda}k'_\sigma k_\lambda\right)\left(\epsilon_{\mu\nu\delta\rho}q^\delta S^\rho\right) &= Q^2\left[\cos\alpha\,(E+E'\cos\theta)+\sin\alpha\cos\phi E'\sin\theta\right]\\ \left(\epsilon^{\mu\nu\sigma\lambda}k'_\sigma k_\lambda\right)\left(\epsilon_{\mu\nu\delta\rho}q^\delta\left(p\cdot qS^\rho-S\cdot qp^\rho\right)\right) &= Q^2M\left[-Q^2\cos\alpha+2EE'\sin\alpha\cos\phi\sin\theta\right]\end{aligned}$$

<u>Homework:</u> Verify these two relations by using the following identity:

$$\epsilon^{\mu\nu\sigma\lambda}\,\epsilon_{\mu\nu\delta\rho}=-2\left(g^\sigma_{\ \delta}\,g^\lambda_{\ \rho}-g^\sigma_{\ \rho}\,g^\lambda_{\ \delta}\right)$$

Then we finally obtain the polarized part of the cross section as

$$\begin{aligned}\frac{\mathrm{d}\sigma^A}{\mathrm{d}E'\,\mathrm{d}\Omega'} &= -\frac{2\alpha^2E'}{MQ^2E}\left\{MG_1(x,Q^2)\left[\cos\alpha\,(E+E'\cos\theta)+\sin\alpha\cos\phi E'\sin\theta\right]\right.\\ &+\left. G_2\left[-Q^2\cos\alpha+2EE'\sin\alpha\cos\phi\sin\theta\right]\right\} \qquad (11.35)\end{aligned}$$

If the initial <u>proton</u> is unpolarized, then we must average over the two possible spin directions along the axis shown in the above figure (i.e., sum the expressions for the angles α and $(\alpha+\pi)$ and divide by 2). Then (11.35) vanishes, and only the unpolarized part in (11.31) remains. If the initial <u>electron</u> is unpolarized, we have to average over the 2 helicity states (i.e., sum the cross section σ^+ and σ^- in (11.31) and divide by 2). Also in this case, only the unpolarized part in (11.31) remains. Therefore, in order to measure G_1 and G_2, <u>both</u> the electron and the proton must be polarized. Most experiments use either longitudinal proton polarization ($\alpha=0,\pi$) or transverse polarization ($\alpha=\pi/2,3\pi/2$).

Experimental data for the structure functions will be introduced in a later lecture.