13 Derivation of the parton model

Remember (Sect. 11):

The cross section for inelastic inclusive electron-proton scattering was expressed in Eq.(11.1) of Sect. 11, in terms of the hadronic tensor $W^{\mu\nu}$, which is given by the current - current correlation function, see Eq.(11.9) of Sect. 11.

Here we calculate $W^{\mu\nu}$ in the quark model, and derive the parton model formulas for the structure functions given in Sect. 12. We will use the method of Feynman diagrams, and therefore we first express the usual product of current operators in Eq.(11.9) of Sect. 9 by the time-ordered product (T - product) of current operators. Then we can use Wick's theorem and Feynman diagrams [1].

(1) We first show the following identity:

$$\int \mathrm{d}^4 z\, e^{iq\cdot z}\, \langle p|J^\mu(z)\, J^\nu(0)|p\rangle = 2\,\mathrm{Im}\left\{ i \int \mathrm{d}^4 z\, e^{iq\cdot z}\, \langle p|T\left(J^\mu(z)\, J^\nu(0)\right)|p\rangle \right\} \tag{13.1}$$

Here the J's are current operators ($\hat{J}$ of Sect. 11), and the T -product is defined by

$$T\left(J^\mu(z)\, J^\nu(0)\right) = \Theta(z^0)\, J^\mu(z)\, J^\nu(0) + \Theta(-z^0)\, J^\nu(0)\, J^\mu(z) \tag{13.2}$$

Here $\Theta(x)$ is the usual step function: $\Theta(x) = 1$ if $x > 0$, and $\Theta(x) = 0$ if $x < 0$.

Proof of (13.1): (i) Consider the l.h.s. of (13.1). We insert a complete set of hadronic states ($\sum_n |n\rangle\langle n| = 1$) between the current operators, and use the z - dependence of the transition matrix elements (see also Eq.(11.4) of Sect. 11), like

$$\langle p|J^\mu(z)|n\rangle = e^{i(p-p_n)\cdot z}\, \langle p|J^\mu(0)|n\rangle \tag{13.3}$$

Then we can do the z -integration, and get

$$\int \mathrm{d}^4 z\, e^{iq\cdot z}\, \langle p|J^\mu(z)\, J^\nu(0)|p\rangle = \sum_n\, (2\pi)^4\, \delta^{(4)}(p+q-p_n)\, \langle p|J^\mu(0)|n\rangle\, \langle n|J^\nu(0)|p\rangle \tag{13.4}$$

[1] A direct evaluation of Eq.(11.9), without Feynman diagrams, is also possible: See lecture of R.L. Jaffe, "Deep inelastic scattering with application to nuclear targets", in: *Relativistic dynamics and quark-nuclear physics*, ed. M.B. Johnson, A. Picklesimer, Wiley, New York, 1986, p. 537.

(ii) Consider the r.h.s. of (13.1). We use the definition of the T-product Eq.(13.2), and insert a complete set of hadronic states between the current operators. Using (13.3) we obtain

$$2\,\mathrm{Im}\left[i\int \mathrm{d}^4z\, e^{iq\cdot z}\,\langle p|T\left(J^\mu(z)\,J^\nu(0)\right)|p\rangle\right] = 2\,\mathrm{Im}\{i\int \mathrm{d}^4z\, e^{iq\cdot z}$$
$$\times\sum_n\Big(\Theta(z_0)\,e^{i(p-p_n)\cdot z}\,e^{-\epsilon z^0}\,\langle p|J^\mu(0)|n\rangle\,\langle n|J^\nu(0)|p\rangle+\Theta(-z_0)\,e^{-i(p-p_n)\cdot z}\,e^{\epsilon z^0}\,\langle p|J^\nu(0)|n\rangle\,\langle n|J^\mu(0)|p\rangle\Big)\} \tag{13.5}$$

Here we introduced convergence factors ($\epsilon\to 0^+$) for the z^0 integrals. The $\int \mathrm{d}^3z$ integration gives 3-momentum conserving δ - functions, and (13.5) becomes

$$2\,(2\pi)^3\;\mathrm{Im}\;i\sum_n\{[\delta^{(3)}(\vec q+\vec p-\vec p_n)\int_0^\infty \mathrm{d}z^0\,e^{i(E_p-E_n+q_0+i\epsilon)z^0}\langle p|J^\mu(0)|n\rangle\,\langle n|J^\nu(0)|p\rangle$$
$$+\delta^{(3)}(-\vec q+\vec p-\vec p_n)\int_{-\infty}^0 \mathrm{d}z^0\,e^{-i(E_p-E_n-q_0+i\epsilon)z^0}\langle p|J^\nu(0)|n\rangle\,\langle n|J^\mu(0)|p\rangle]\}$$

Performing the z^0 integral, this becomes:

$$-2\,\mathrm{Im}\sum_n\,(2\pi)^3\;[\delta^{(3)}(\vec q+\vec p-\vec p_n)\,\frac{\langle p|J^\mu(0)|n\rangle\,\langle n|J^\nu(0)|p\rangle}{E_p-E_n+q_0+i\epsilon}$$
$$+\delta^{(3)}(-\vec q+\vec p-\vec p_n)\,\frac{\langle p|J^\nu(0)|n\rangle\,\langle n|J^\mu(0)|p\rangle}{E_p-E_n-q_0+i\epsilon}] \tag{13.6}$$

We now use the Cauchy's formula [2] (for $\epsilon\to 0^+$)

$$\frac{1}{A+i\epsilon}=P\,\frac{1}{A}-i\pi\delta(A) \tag{13.7}$$

where P means the principal value. The imaginary part of (13.6) comes from the second term in (13.7). Noting that $E_p-E_n<0$ (because the proton state with momentum p is the lowest energy state for given baryon number $=1$) and $q_0>0$ (the electron looses energy in the inelastic scattering process), we see that the denominator of the second term in (13.6) cannot vanish, and therefore this term gives no contribution to the imaginary part. Then (13.6) becomes finally

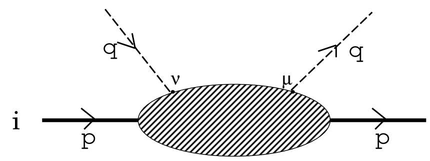
$$\sum_n\,(2\pi)^4\;\delta^{(4)}(p+q-p_n)\,\langle p|J^\mu(0)|n\rangle\,\langle n|J^\nu(0)|p\rangle \tag{13.8}$$

which is equal to (13.4). This concludes the proof of Eq.(13.1).

[2]An intuitive check of the δ-function term in this formula is as follows: $\frac{1}{A+i\epsilon}=\frac{A-i\epsilon}{A^2+\epsilon^2}$. The imaginary part of this is equal to $-\frac{\epsilon}{A^2+\epsilon^2}$. If $A\neq 0$, this is zero in the limit $\epsilon\to 0^+$, but if $A=0$, this is $-\infty$ in the limit $\epsilon\to 0^+$. This is just the property of $-\delta(A)$. The precise relation is $\frac{\epsilon}{A^2+\epsilon^2}=\pi\,\delta(A)$ in the limit $\epsilon\to 0^+$.

The amplitude for forward Compton scattering on the proton (forward scattering amplitude of proton (momentum p) and virtual photon (momentum q)) is defined as [3]

$$T^{\mu\nu}(p,q) = i\,2E_p \int \mathrm{d}^4 z\, e^{iq\cdot z} \left\langle p \right| T\left(J^\mu(z)\, J^\nu(0)\right) \left| p \right\rangle \tag{13.9}$$

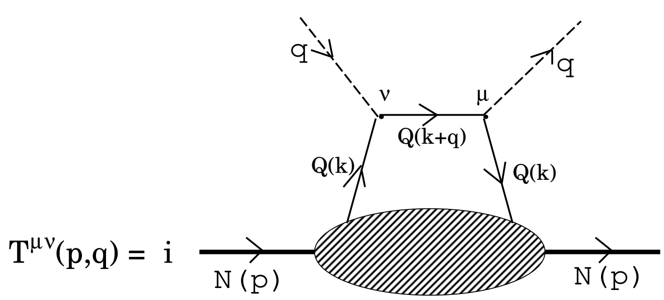


Then, from Eq.(11.9) of Sect. 11, and Eq.(13.1), the hadronic tensor can be expressed by the imaginary part of $T^{\mu\nu}$:

$$W^{\mu\nu} = \frac{1}{2\pi}\,\mathrm{Im}\, T^{\mu\nu} \tag{13.10}$$

This is the fundamental relation to calculate the hadronic tensor (and structure functions) in any model.

(2) Now we calculate the Compton amplitude in the quark model. As in Sect. 12, we rely on our intuition on the Daruma Otoshi, i.e., in the high energy process only one quark (the "active quark") will contribute to the scattering, while the other 2 quarks do nothing ("spectator quarks"). The Compton amplitude is then expressed by the following diagram ("handbag diagram"):



Here $N(p)$ means a nucleon (proton) with momentum p, $Q(k)$ means the active quark inside the proton with momentum k, and the shaded area indicates the propagation of the 2 spectator quarks (including also sea quarks and interactions). There is a loop in this diagram, and therefore we must integrate over the momentum k.

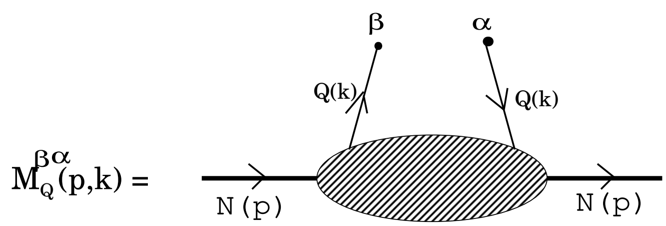
[3]The factor $2E_p$ arises from our non-covariant normalization of state vectors, as explained in Sect. 10.

[Note for specialists: According to Feynman rules, each quark line is translated to iS, where S is the usual Feynman propagator introduced in Sect.13 of the spring semester. Because $i^4 = 1$, we do not attach i to propagators in the following.]

Translating this diagram into a formula, we get

$$T^{\mu\nu}(p,q) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{i=u,d} \mathrm{Tr}\left[M_i(p,k)\, t_i^{\mu\nu}(k,q)\right] \tag{13.11}$$

Here $i = u, d$ labels the flavor of the active quark, Tr denotes the trace over Dirac indices, and $M_i(p,k)$ corresponds to the lower part of the handbag diagram:

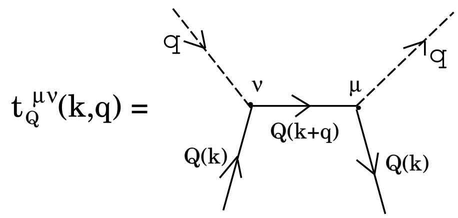


In this diagram, the external quark lines (momentum k) represent Feynman propagators, and the external nucleon line (momentum p) represents the Dirac spinor of the nucleon ($u(\vec{p}, S)$). Because the Feynman propagator is a Dirac matrix (remember Sect. 13 of the spring semester), $M_i(p,k)$ is also a Dirac matrix, as indicated by the Dirac indices α and β. The quantity $M_i(p,k)$ can be considered as the propagator (2-point function) of the quark $i = u, d$ inside the nucleon. It is defined by (see the previous footnote for the origin of the factor $2E_p$)

$$M_i^{\beta\alpha}(p,k) = 2E_p \int \mathrm{d}^4 z\, e^{ik\cdot z}\, \langle p| T\left(\overline{\psi}_\alpha(0)\, \psi_\beta(z)\right) |p\rangle \tag{13.12}$$

Here the ψ's are the field operators for the active quark with flavor $i = u, d$ (in second quantization), i.e., $\overline{\psi}_\alpha(0)$ creates the quark i at position $z = 0$, and $\psi_\beta(z)$ annihilates the quark i at position z. By using methods explained in the next lecture, one can show that the part of (13.12), which contributes to the structure functions in the Bjorken limit, is real. Therefore, in the following we simply assume that $M_i(p,k)$ is real.

The quantity $t_i^{\mu\nu}(p,k)$ in Eq.(13.11) corresponds to the upper part of the handbag diagram:



Here the external quark propagators (momentum k) are not included (because they have been included already in the definition of M_i).
$t_i^{\mu\nu}(k,q)$ is the amplitude for forward Compton scattering on the quark $i = u, d$, and is given by

$$t_i^{\mu\nu}(k,q) = (ie_i)^2\,\gamma^\mu\,S(k+q)\,\gamma^\nu = -e_i^2\,\gamma^\mu\,\frac{\not{k}+\not{q}+m_i}{(k+q)^2 - m_i^2 + i\epsilon}\,\gamma^\nu \tag{13.13}$$

where we used the form of the Feynman propagator given in Sect. 13 of the spring semester.
[Note for specialists: According to Feynman rules, a quark-photon vertex is translated into $(-ie_i\gamma^\mu)$.]

The imaginary part of (13.13) is given by (see Eq.(13.7))

$$\mathrm{Im}\,t_i^{\mu\nu}(k,q) = \pi\,e_i^2\,\delta\left((k+q)^2 - m_i^2\right)\right)\gamma^\mu\,(\not{k}+\not{q}+m_i)\,\gamma^\nu \tag{13.14}$$

Here we use

$$(k+q)^2 - m_i^2 = k^2 + q^2 + 2k\cdot q - m_i^2 = 2p\cdot q\left(-x + \frac{k\cdot q}{p\cdot q} + \frac{k^2 - m_i^2}{2\,p\cdot q}\right)$$

where $x = -q^2/(2p\cdot q)$ is the Bjorken variable for the nucleon. Then (13.14) becomes

$$\mathrm{Im}t_i^{\mu\nu}(k,q) = \frac{\pi\,e_i^2}{2p\cdot q}\,\delta\left(x - \frac{k\cdot q}{p\cdot q} - \frac{k^2 - m_i^2}{2\,p\cdot q}\right)\gamma^\mu\,(\not{k}+\not{q}+m_i)\,\gamma^\nu \tag{13.15}$$

The Bjorken limit (B.L., see Sect. 12) means $Q^2 = -q^2 \to \infty$ and $p\cdot q \to \infty$ such that the ratio $x = Q^2/(2p\cdot q)$ is fixed. (Note that $0 < x < 1$). In this limit (13.15) simplifies to

$$\mathrm{Im}\,t_i^{\mu\nu}(k,q) \xrightarrow{\text{B.L.}} \frac{\pi\,e_i^2}{2p\cdot q}\,\delta\left(x - \frac{k\cdot q}{p\cdot q}\right)\gamma^\mu\,\not{q}\,\gamma^\nu \tag{13.16}$$

The hadronic tensor is given by $W^{\mu\nu} = \frac{1}{2\pi}\,\mathrm{Im}\,T^{\mu\nu}$ (see Eq.(13.10)), and from (13.11) and (13.16) we obtain in the quark model

$$W^{\mu\nu}(p,q) = \frac{1}{4p\cdot q}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_i e_i^2\,\delta\left(x - \frac{k\cdot q}{p\cdot q}\right)\,\mathrm{Tr}\left[M_i(p,k)\,\gamma^\mu\,\not{q}\,\gamma^\nu\right] \tag{13.17}$$

Use $\not{q} = \gamma^\lambda q_\lambda$, and the following formula for the product of three Dirac γ-matrices:

$$\gamma^\mu \gamma^\lambda \gamma^\nu = \left(g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda} - g^{\mu\nu} g^{\lambda\sigma}\right) \gamma_\sigma - i\epsilon^{\mu\lambda\nu\sigma} \gamma_\sigma \gamma_5 \tag{13.18}$$

Then we can separate the symmetric and antisymmetric parts of the hadronic tensor (13.17):

$$W^{\mu\nu}(p,q) = W_s^{\mu\nu} + W_a^{\mu\nu} \tag{13.19}$$

where

$$\begin{aligned} W_s^{\mu\nu}(p,q) &= \frac{q_\lambda}{4p\cdot q} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_i e_i^2 \, \delta\left(x - \frac{k\cdot q}{p\cdot q}\right) \left(g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda} - g^{\mu\nu} g^{\lambda\sigma}\right) \\ &\times \ \mathrm{Tr}\left[M_i(p,k)\, \gamma_\sigma\right] \end{aligned} \tag{13.20}$$

$$W_a^{\mu\nu}(p,q) = i\epsilon^{\mu\nu\lambda\sigma} \frac{q_\lambda}{4p\cdot q} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_i e_i^2 \, \delta\left(x - \frac{k\cdot q}{p\cdot q}\right) \mathrm{Tr}\left[M_i(p,k)\, \gamma_\sigma \gamma_5\right] \tag{13.21}$$

We now consider each part separately:

- The symmetric part $W_s^{\mu\nu}$ has been parametrized in terms of the structure functions F_1 and F_2, see Eq.(11.15) of Sect.11:

$$W_s^{\mu\nu}(p,q) = F_1(x,Q^2)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + F_2(x,Q^2)\, \frac{\tilde{p}^\mu \tilde{p}^\nu}{p\cdot q} \tag{13.22}$$

  where we defined the 4-vector $\tilde{p}^\mu$ by

$$\tilde{p}^\mu \equiv p^\mu - \frac{p\cdot q}{q^2}\, q^\mu \tag{13.23}$$

  such that $\tilde{p}\cdot q = 0$. By comparing the coefficients of $g^{\mu\nu}$ in (13.20) and (13.22), we can obtain F_1 as

$$F_1(x,Q^2) = \frac{1}{4(p\cdot q)} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_i e_i^2 \, \delta\left(x - \frac{k\cdot q}{p\cdot q}\right) \mathrm{Tr}\left[M_i(p,k)\, \not{q}\right] \tag{13.24}$$

  In order to get also F_2, we note that the trace (contraction of μ and ν) of the Lorentz tensor $(\ldots)$ in (13.20) is equal to

$$g^{\mu\lambda} g_\mu{}^\sigma + g^{\mu\sigma} g_\mu{}^\lambda - 4g^{\lambda\sigma} = 2g^{\lambda\sigma} - 4g^{\lambda\sigma} = -2g^{\lambda\sigma}$$

  Comparing this with the original form (13.20), we can say the following:

The trace $(W_s)^\mu{}_\mu$ must be equal to twice the coefficient of $g^{\mu\nu}$. (*)

From (13.22), this means that

$$(W_s)^\mu{}_\mu = -3F_1(x,Q^2) + \frac{\tilde{p}^2}{p\cdot q}F_2(x,Q^2) = -2F_1(x,Q^2) \tag{13.25}$$

where the first equality follows from (13.22) and the second equality from the statement (*). In the Bjorken limit, it follows from the definition of $\tilde{p}^\mu$ (see Eq.(13.23)) and the definition of the variable $x = -q^2/(2p\cdot q)$, that $\tilde{p}^2/(p\cdot q) = 1/(2x)$. By using this relation in (13.25), we can obtain F_2 as

$$\begin{aligned} F_2(x,Q^2) &= 2xF_1(x,Q^2) && (13.26)\\ &= \frac{x}{2(p\cdot q)}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_i e_i^2\,\delta\left(x-\frac{k\cdot q}{p\cdot q}\right)\mathrm{Tr}\left[M_i(p,k)\,\not{q}\right] && (13.27)\end{aligned}$$

We can express the results (13.24) and (13.27) in the following way:

$$F_1(x,Q^2) = \frac{1}{2}\sum_i e_i^2\,q_i(x)\,, \qquad F_2(x,Q^2) = x\sum_i e_i^2\,q_i(x) \tag{13.28}$$

where we defined the "spin independent quark distribution function" $q_i(x)$ by

$$q_i(x) = \frac{1}{2(p\cdot q)}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\delta\left(x-\frac{k\cdot q}{p\cdot q}\right)\mathrm{Tr}\left[M_i(p,k)\,\not{q}\right] \tag{13.29}$$

- The antisymmetic part $W_a^{\mu\nu}$ has been parametrized in terms of the structure functions g_1 and g_2, see Eq.(11.16) of Sect. 11:

$$W_a^{\mu\nu} = \frac{M}{p\cdot q}\,i\epsilon^{\mu\nu\lambda\sigma}\,q_\lambda\left[g_1(x,Q^2)\,S_\sigma + g_2(x,Q^2)\left(S_\sigma - \frac{(S\cdot q)\,p_\sigma}{p\cdot q}\right)\right] \tag{13.30}$$

Comparison with the quark model expression Eq.(13.21) gives immediately

$$\begin{aligned} S_\sigma\,g_1(x,Q^2) &= \frac{1}{4M}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_i e_i^2\,\delta\left(x-\frac{k\cdot q}{p\cdot q}\right)\mathrm{Tr}\left[M_i(p,k)\,\gamma_\sigma\gamma_5\right]\\ &\equiv S_\sigma\,\frac{1}{2}\sum_i e_i^2\,\Delta q_i(x) && (13.31)\\ g_2(x,Q^2) &= 0 && (13.32)\end{aligned}$$

In (13.31) we defined the "spin-dependent quark distribution function" by the following relation:

$$S^\mu\,\Delta q_i(x) = \frac{1}{2M}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\delta\left(x-\frac{k\cdot q}{p\cdot q}\right)\mathrm{Tr}\left[M_i(p,k)\,\gamma^\mu\,\gamma_5\right] \tag{13.33}$$

In order to express the result (13.33) similar to Eq.(13.29), we contract both sides of Eq.(13.33) with q_μ, and divide by $(p \cdot q)$:

$$\left(M \frac{S \cdot q}{p \cdot q} \right) \Delta q_i(x) \;=\; \frac{1}{2(p \cdot q)} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_i e_i^2 \, \delta \left(x - \frac{k \cdot q}{p \cdot q} \right) \mathrm{Tr} \left[M_i(p,k) \not{q} \gamma_5 \right] \tag{13.34}$$

Note that both functions (13.29) and (13.34) are defined in the Bjorken limit. In the next lecture, we will see how to calculate the Bjorken limit, and why one can interpret (13.29) and (13.34) as "quark distribution functions".

Here we explain only the result:

Consider a nucleon with 4-momentum p^μ and spin direction parallel (positive helicity) or anti-parallel (negative helicity) to its 3-momentum. Denote the probability to find a quark, with flavor $i = u, d$, momentum fraction x, and spin direction parallel (↑) or anti-parallel (↓) to the nucleon spin, by $q_i^\uparrow(x)$ and $q_i^\downarrow(x)$. Then the functions (13.29) and (13.34) can be expressed as

$$q_i(x) \;=\; q_i^\uparrow(x) + q_i^\downarrow(x) \tag{13.35}$$

$$\Delta q_i(x) \;=\; q_i^\uparrow(x) - q_i^\downarrow(x) \tag{13.36}$$

From this interpretation, and the (naive) assumption that the nucleon consists only of 3 valence quarks, it follows that

$$\int_0^1 \mathrm{d}x \sum_i q_i(x) = 3 \,, \qquad \int_0^1 \mathrm{d}x \, x \sum_i q_i(x) = 1 \,, \qquad \int_0^1 \mathrm{d}x \sum_i \Delta q_i(x) = 1 \tag{13.37}$$

Here the first relation ("number sum rule") means the the nucleon consists of 3 valence quarks, the second relation ("momentum sum rule") means that the 3 valence quark carry 100% of the nucleon momentum, and the third relation ("spin sum rule") means that they carry 100% of the nucleon spin.

However, in the nucleon there are also "sea quarks" ($q\overline{q}$ pairs) and gluons, which carry no baryon number but can carry momentum and spin. Therefore, in the extended parton model, the number sum rule remains true for the valence quarks, but the momentum sum rule and the spin sum rule will be modified.

More details will be explained in the next lecture.