14 Physical meaning of structure functions and quark distribution functions

Remember (Sec. 13): (1) The spin-independent structure functions $F_1(x)$, $F_2(x)$ in the Bjorken limit are given by $F_1(x) = \frac{1}{2}\sum_{i=u,d} e_i^2\, q_i(x)$ and $F_2(x) = x\sum_{i=u,d} e_i^2\, q_i(x)$. Here the spin-independent quark distribution function $q_i(x)$ is defined by Eq.(13.29). If we insert here Eq.(13.12) for $M_i(p,k)$, we obtain the more explicit form [1]

$$q_i(x) = \frac{E_p}{p\cdot q}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\int \mathrm{d}^4 z\, e^{ik\cdot z}\,\delta\left(x - \frac{k\cdot q}{p\cdot q}\right)\langle p,S|T\left(\overline{\psi}_i(0)\,\not{q}\,\psi_i(z)\right)|p,S\rangle \tag{14.1}$$

(2) The spin-dependent structure functions $g_1(x)$, $g_2(x)$ in the Bjorken limit are given by $g_1(x) = \frac{1}{2}\sum_{i=u,d} e_i^2\,\Delta\, q_i(x)$ and $g_2(x) = 0$. Here the spin-dependent quark distribution function $\Delta q_i(x)$ is defined by Eq.(13.34). If we insert here Eq.(13.12) for $M_i(p,k)$, we obtain the more explicit form

$$\left(M\frac{S\cdot q}{p\cdot q}\right)\Delta\, q_i(x) = \frac{E_p}{p\cdot q}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\int \mathrm{d}^4 z\, e^{ik\cdot z}\,\delta\left(x - \frac{k\cdot q}{p\cdot q}\right)\langle p,S|T\left(\overline{\psi}_i(0)\,\not{q}\,\gamma_5\,\psi_i(z)\right)|p,S\rangle \tag{14.2}$$

Why can we call (14.1) and (14.2) "quark distribution functions"? Let us first check the normalizations: If we integrate over x, the delta functions in (14.1) and (14.2) go away. Then the integral $\frac{\mathrm{d}^4 k}{(2\pi)^4}$ gives $\delta^{(4)}(z)$, and we obtain

$$\int_0^1 \mathrm{d}x\, q_i(x) \;=\; \frac{E_p}{p\cdot q}\langle p,S|\overline{\psi}_i(0)\,\gamma^\mu\,\psi_i(0)|p,S\rangle\, q_\mu \tag{14.3}$$

$$\left(M\frac{S\cdot q}{p\cdot q}\right)\int_0^1 \mathrm{d}x\,\Delta\, q_i(x) \;=\; \frac{E_p}{p\cdot q}\langle p,S|\overline{\psi}_i(0)\,\gamma^\mu\,\gamma_5\,\psi_i(0)|p,S\rangle\, q_\mu \tag{14.4}$$

where we left out the time-ordering symbol T, because both quark field operators now appear at the same space-time point $z=0$. In our non-covariant normalization of the state $|p,S\rangle$ we have the following relations [2] :

$$\langle p,S|\overline{\psi}_i(0)\,\gamma^\mu\,\psi_i(0)|p,S\rangle \;=\; N_i\left(\frac{M}{E_p}\overline{u}(\vec{p},S)\,\gamma^\mu\, u(\vec{p},S)\right) = N_i\,\frac{p^\mu}{E_p} \tag{14.5}$$

$$\langle p,S|\overline{\psi}_i(0)\,\gamma^\mu\,\gamma_5\,\psi_i(0)|p,S\rangle \;=\; (\Delta N_i)\left(\frac{M}{E_p}\overline{u}(\vec{p},S)\,\gamma^\mu\,\gamma_5\, u(\vec{p},S)\right) = (\Delta N_i)\,\frac{M}{E_p}\,S^\mu \tag{14.6}$$

[1] In this Section, we also attach the flavor label to the quark fields (ψ_i), in order to make the flavor dependence explicit. Also, we write $|p,S\rangle$ for the nucleon state vectors in order to indicate the spin direction more explicitly.

[2] To derive those relations one needs second quantization, but they can be understood intuitively as follows: For relation (14.5), consider the case $\mu=0$, where $\psi_i^\dagger\psi_i$ is the operator for the number density of quarks with flavor i. Then $\langle p,S|\psi_i^\dagger\,\psi_i|p,S\rangle = \frac{N_i}{V}\langle p,S|p,S\rangle = N_i$, where we used our non-covariant normalization $\langle p,S|p,S\rangle = V$, see Sect. 10. In a similar way, for relation (14.6), consider the case $\mu = 1,2,3$, where $\overline{\psi}_i\vec{\gamma}\gamma_5\,\psi_i$ is the operator for the spin density of quarks with flavor i, i.e.; (the number density of quarks with spin parallel to the nucleon spin) minus (the number density of quarks with spin anti-parallel to the nucleon spin). Its expectation value is given by $\langle p,S|\overline{\psi}_i^\dagger\,\vec{\gamma}\gamma_5\,\psi_i|p,S\rangle = \frac{\Delta N_i}{V}\,\frac{M}{E_p}\,\vec{S}\,\langle p,S|p,S\rangle = \Delta N_i\,\frac{M}{E_p}\,\vec{S}$.

where $N_i \equiv N_i^{\uparrow} + N_i^{\downarrow}$ is the number of quarks (of either spin direction) with flavor $i = u, d$ inside the hadron (proton), and $\Delta N_i \equiv N_i^{\uparrow} - N_i^{\downarrow}$ is : (the number of quarks with spin parallel to the nucleon spin) minus (the number of quarks with spin anti-parallel to the nucleon spin) with flavor $i = u, d$ inside the hadron. In the last step of Eq.(14.6), we used Eq.(9.7) of the spring semester for the spin 4-vector S^μ (except that now we do not have a factor $\frac{1}{2}$ in the definition of S^μ). We then obtain from Eq.(14.3) and (14.4):

$$\int_0^1 \mathrm{d}x \, q_i(x) = N_i = N_i^{\uparrow} + N_i^{\downarrow} \tag{14.7}$$

$$\int_0^1 \mathrm{d}x \, \Delta q_i(x) = \Delta N_i = N_i^{\uparrow} - N_i^{\downarrow} \tag{14.8}$$

The relation (14.7) suggests that $q_i(x)$ may be the sum of probability densities to find a quark (flavor i) with spin parallel ($\uparrow$) and anti-parallel ($\downarrow$) to the nucleon spin, with some value of x. Similarly, relation(14.8) suggests that $\Delta q_i(x)$ may be the difference of probability densities to find a quark (flavor i) with spin parallel ($\uparrow$) and anti-parallel ($\downarrow$) , to the nucleon spin, with some value of x.

In order to see the meaning of the variable x, we define light-cone momentum components as follows:

$$k^{\pm} \equiv \frac{1}{\sqrt{2}} \left(k^0 \pm k^3\right) , \qquad \vec{k}_T = \left(k^1, \, k^2\right) \tag{14.9}$$

This is a simple variable transformation from the Minkowski 4-vector components $k^\mu = (k^0, k^1, k^2, k^3)$ to the light-cone 4-vector components $k^\mu = (k^+, k^-, k^1, k^2)$. We also define $k_\pm \equiv \frac{1}{\sqrt{2}} (k_0 \pm k_3)$, so that $k^+ = k_-$ and $k^- = k_+$. The scalar product of two 4-vectors a and b can be rewritten in terms of light-cone components by

$$a \cdot b = a^0 \, b^0 - \vec{a} \cdot \vec{b} = a^+ \, b^- + a^- \, b^+ - \vec{a}_T \cdot \vec{b}_T \tag{14.10}$$

We will also define the light-cone plus (+) and minus (-) components of the Dirac γ-matrices as

$$\gamma^{\pm} = \frac{1}{\sqrt{2}} \left(\gamma^0 \pm \gamma^3\right) , \qquad \gamma_{\pm} = \frac{1}{\sqrt{2}} \left(\gamma_0 \pm \gamma_3\right)$$

so that $\gamma^+ = \gamma_-$ and $\gamma^- = \gamma_+$.

We now choose a frame where the momentum transfer from the electron to the nucleon is along the $-\hat{z}$ direction: In usual coordinates, $q^\mu = (q^0, 0, 0, q^3)$ with $q^3 < 0$. Then the Bjorken limit (B.L.)

corresponds to the limit $q^0 \to \infty$, $q^3 \to -\infty$, such that $q^- = (q^0 - q^3)/\sqrt{2} \to \infty$ and $q^+ = (q^0 + q^3)\sqrt{2}$ is finite, because in this case the original definitions of the B.L. are satisfied:

$$q^2 = 2q^+ \, q^- \xrightarrow{B.L.} \infty \,, \qquad x = \frac{-q^2}{2p \cdot q} = \frac{-q^+ \, q^-}{p^+ \, q^- + p^- \, q^+} \xrightarrow{B.L.} -\frac{q^+}{p^+} = \text{finite} \tag{14.11}$$

From the second relation in (14.11) we see that, in the Bjorken limit, $q^+ \xrightarrow{B.L.} -p^+ \, x = -\frac{1}{\sqrt{2}} \left(E_p + p^3\right) \, x$. For example, in the rest frame of the hadron, this is equal to $-(Mx)/\sqrt{2}$.

Now we can express the functions $q_i(x)$ of Eq.(14.1) and $\Delta q_i(x)$ of Eq.(14.2) in terms of light-cone coordinates. For this, we note the following most important relation in the Bjorken limit:

$$\frac{k \cdot q}{p \cdot q} = \frac{k^+ \, q^- + k^- \, q^+}{p^+ \, q^- + p^- \, q^+} \xrightarrow{B.L.} \frac{k^+}{p^+} = x \tag{14.12}$$

This means that x is really a momentum fraction carried by the quark, but not of a usual momentum component (x, y, z component), but of a light-cone momentum component: $k^+ = p^+ x$. For example, in the rest frame of the hadron, $k^+ = (Mx)\sqrt{2}$.
For the other factors in Eqs.(14.1) and (14.2), we use the relation

$$\frac{\not{q}}{p \cdot q} = \frac{\gamma^+ \, q^- + \gamma^- \, q^+}{p^+ \, q^- + p^- \, q^+} \xrightarrow{B.L.} \frac{\gamma^+}{p^+}$$

Taking also the Bjorken limit on the l.h.s. of Eq.(14.2) gives $(S \cdot q)/(p \cdot q) \xrightarrow{B.L.} S^+/p^+$. Then the functions $q_i(x)$ and $\Delta q_i(x)$ of Eq.(14.1) and (14.2) can be expressed as

$$q_i(x) \;=\; \frac{E_p}{p^+} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \delta\left(x - \frac{k^+}{p^+}\right) \int \mathrm{d}^4 z \, e^{ik \cdot z} \langle p, S | T\left(\overline{\psi}_i(0) \, \gamma^+ \, \psi_i(z)\right) | p, S \rangle \tag{14.13}$$

$$\left(\frac{MS^+}{p^+}\right) \Delta q_i(x) \;=\; \frac{E_p}{p^+} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \delta\left(x - \frac{k^+}{p^+}\right) \int \mathrm{d}^4 z \, e^{ik \cdot z} \langle p, S | T\left(\overline{\psi}_i(0) \, \gamma^+ \, \gamma_5 \, \psi_i(z)\right) | p, S \rangle \tag{14.14}$$

Note that the δ-function in those expressions fixes the light-cone momentum component k^+ of the quark as $k^+ = p^+ \, x$. Using here $k \cdot z = k^+ z^- + k^- z^+ - \vec{k}_T \cdot \vec{z}_T$, we can integrate over k^- and $\vec{k}_T$:

$$\int \frac{\mathrm{d}k^- \, \mathrm{d}^2 k_T}{(2\pi)^3} e^{i(k^- \, z^+ - \vec{k}_T \cdot \vec{z}_T)} = \delta(z^+) \, \delta^{(2)}(\vec{z}_T) \tag{14.15}$$

and Eqs.(14.13) and (14.14) become

$$q_i(x) \;=\; E_p \int \frac{\mathrm{d}z^-}{2\pi} \, e^{(p^+ x) z^-} \, \langle p, S | T\left(\overline{\psi}_i(0) \, \gamma^+ \, \psi_i(z^-)\right) | p, S \rangle \tag{14.16}$$

$$\left(\frac{MS^+}{p^+}\right) \Delta q_i(x) \;=\; E_p \int \frac{\mathrm{d}z^-}{2\pi} \, e^{(p^+ x) z^-} \, \langle p, S | T\left(\overline{\psi}_i(0) \, \gamma^+ \, \gamma_5 \, \psi_i(z^-)\right) | p, S \rangle \tag{14.17}$$

where $\psi_i(z^-) \equiv \psi_i(z^+ = 0, z^-, \vec{z}_T = \vec{0}_T)$. [3] The quark field operators in (14.16) and (14.17) are separated by a distance on the light cone, because $z^+ = 0$ means that $z^0 + z^3 = 0$. Therefore (14.16) and (14.17) can be called light-cone correlation functions. The time-ordering symbol T in those relations is actually unnecessary. To see this, we note that, for $z^+ = 0$ but finite $\vec{z}_T$, the usual time variable is given by $z^0 = z^-/\sqrt{2}$. Therefore, denoting the Dirac indices of the quark field operator by α, $\beta = 1, \ldots 4$, we have for $z^+ = 0$ [4]:

$$\begin{aligned} T\left(\overline{\psi}^{(\alpha)}(0)\,\psi^{(\beta)}(z^-, \vec{z}_T)\right) &= \theta(-z^-)\;\overline{\psi}^{(\alpha)}(0)\,\psi^{(\beta)}(z^-, \vec{z}_T) - \theta(z^-)\;\psi^{(\beta)}(z^-, \vec{z}_T)\,\overline{\psi}^{(\alpha)}(0) \\ &= \overline{\psi}^{(\alpha)}(0)\,\psi^{(\beta)}(z^-, \vec{z}_T) - \theta(z^-)\,\{\overline{\psi}^{(\alpha)}(0)\,, \psi^{(\beta)}(z^-, \vec{z}_T)\}_+ \\ &= \overline{\psi}^{(\alpha)}(0)\,\psi^{(\beta)}(z^-, \vec{z}_T) \end{aligned}$$

where $\{\ldots\}_+$ denotes the anticommutator. (For any 2 operators a, B, the anticommutator is defined by $\{A, B\}_+ \equiv AB + BA$.) In the second step of the above relation we used $\theta(-z^0) = 1 - \theta(z^0)$, and in the last step we used that the anticommutator between fermion field operators separated by a space-like distance $z^2 = 2z^+z^- - \vec{z}_T^2 = -\vec{z}_T^2 < 0$ vanishes because of causality [5]. Then (14.16) and (14.17) become finally

$$\begin{aligned} q_i(x) &= E_p \int \frac{\mathrm{d}z^-}{2\pi}\, e^{(p^+x)z^-}\, \langle p, S|\overline{\psi}_i(0)\,\gamma^+\,\psi_i(z^-)|p, S\rangle & (14.18) \\ \left(M\frac{S^+}{p^+}\right)\Delta q_i(x) &= E_p \int \frac{\mathrm{d}z^-}{2\pi}\, e^{(p^+x)z^-}\, \langle p, S|\overline{\psi}_i(0)\,\gamma^+\,\gamma_5\,\psi_i(z^-)|p, S\rangle & (14.19) \end{aligned}$$

Let us check here the sum rules (14.7) and (14.8) again: If we integrate the exponentials in (14.18) and (14.19) over x, we get a factor $(2\pi)\delta(p^+z^-) = \frac{2\pi}{p^+}\delta(z^-)$, and from Eq.(14.5) we have $\langle p, S|\overline{\psi}_i\,\gamma^+\,\psi_i|p, S\rangle = N_i(p^+/E_p)$ and $\langle p, S|\overline{\psi}_i\,\gamma^+\,\gamma_5\psi_i|p, S\rangle = (\Delta N_i)\,\frac{M}{E_p}\,S^+$. Therefore the sum rules (14.7) and (14.8) are satisfied.

[3] We define that the limit $z^+ = 0$ is taken first, and then $\vec{z}_T = 0$. In terms of light-cone momenta, this means that in Eq.(14.15), we first integrate over k^-, keeping the transverse momenta $\vec{k}_T$ fixed, and then integrate over the transverse momenta of the quarks. (It is also possible to keep $\vec{k}_T$ finite, then function $q_i(x, \vec{k}_T)$ is called a "transverse momentum dependent quark distribution function". This is a subject of current experimental and theoretical research , but we cannot consider this case in our lectures.)

[4] Note that the T-product of fermion field operators $A(x)$ and $B(y)$ is generally defined by $T(A(x)B(y)) = \theta(x^0 - y^0)\,A(x)B(y) - \theta(y^0 - x^0)\,B(y)A(x)$, where $\theta(x)$ is the usual step function.

[5] Two events separated by a light-like distance are independent of each other, because no information (not even a light signal) can be exchanged between them. In quantum field theory, this means that the commutator (for boson operators) or the anticommutator (for fermion operators) vanishes if these operators are saparated by a light-like distance.

In order to get a more explicit expression for $\Delta q_i(x)$, we consider the case where the nucleon has definite helicity [6] $\lambda_N = \pm 1$. This means that the spin vector $\vec{S}$, which is a unit vector in the direction of the spin of the nucleon, is given by $\vec{S} = \lambda_N \hat{\vec{p}}$. We insert this relation into the expression for the spin 4-vector, which is given in usual components $\mu = 0,1,2,3$ by (see Eq.(11.14) of Sect. 11)

$$S^\mu = \left(\frac{\vec{p}\cdot\vec{S}}{M},\ \vec{S} + \frac{\vec{p}(\vec{p}\cdot\vec{S})}{M(E_p + M)} \right)$$

to get

$$S^\mu = \lambda_N \left(\frac{p}{M},\ \hat{\vec{p}}\,\frac{E_p}{M} \right) \tag{14.20}$$

where $p \equiv |\vec{p}|$. If we choose the direction $\hat{\vec{p}}$ along the positive z-direction [7] , we obtain for the usual components $\mu = 0,1,2,3$: $S^\mu = \lambda_N \left(\frac{p}{M}, 0, 0, \frac{E_p}{M} \right)$. For the light-cone component this gives $S^+ = \lambda_N \frac{p^+}{M}$, and the factor on the l.h.s. of Eq.(14.19) becomes the helicity of the nucleon: $\left(M \frac{S^+}{p^+} \right) = \lambda_N$. Therefore, for a nucleon with helicity $\lambda_N = \pm 1$ moving in the positive z-direction, we obtain from (14.19):

$$\Delta q_i(x) \quad = \quad \lambda_N\, E_p \int \frac{\mathrm{d}z^-}{2\pi}\, e^{(p^+ x) z^-}\, \langle p, S | \overline{\psi}_i(0)\, \gamma^+\, \gamma_5\, \psi_i(z^-) | p, S \rangle \tag{14.21}$$

15 Field theory on the light cone (an introduction)

So, why are $q_i(x)$ and $\Delta q_i(x)$ of Eqs.(14.18), (14.21) probability distributions? To understand this, we remember that, in usual Minkowski coordinates, the field quantization is done by imposing conditions like

$\{\psi^{(\alpha)}(t,\vec{z}), \psi^{\dagger(\beta)}(t,\vec{z}')\}_+ = \delta^{(3)}(\vec{z} - \vec{z}')$, or $\{\psi^{(\alpha)}(\vec{z},t), \psi^{(\beta)}(\vec{z}',t)\}_+ = 0$ etc, at equal time (t). With the light-cone variables defined by

$$z^\pm = \frac{1}{\sqrt{2}} \left(z^0 \pm z^3 \right), \qquad \vec{z}_T = \left(z^1, z^2 \right)$$

we can impose similar quantization conditions, but we must decide whether we use z^+ or z^- as the "light-cone time". Usually one uses z^+ as the "light-cone time", and performs the field quantization

[6] We note that the function $\Delta q_i(x)$ itself is independent of the choice for the nucleon spin or momentum components. The matrix element on the r.h.s. of Eq.(14.19), however, depends on the choice of the nucleon spin and momentum components, and this dependence is expressed by the factor $\left(M \frac{S^+}{p^+} \right)$ on the l.h.s. of (14.19).

[7] Note that we assumed already that $\vec{q}$ is in the negative z - direction ($q^1 = q^2 = 0, q^3 < 0$), which was just a definition of the z - axis. Then, in the laboratory system ($p = 0$), (14.20) simply means that the target nucleon is polarized along the z - axis. More generally, one can apply a Lorentz transformation from the laboratory system to a system where the initial nucleon and virtual photon move collinear (the nucleon along $\hat{z}$ and the photon along $-\hat{z}$).

at equal values of z^+.

The Lagrangian of the Dirac theory (see Eq.(1.2) of Sect. 1) can easily be expressed by light-cone variables, for example for the free part:

$$\mathcal{L} = \overline{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi = \overline{\psi} \left(i\gamma^+ \partial_+ + i\gamma^- \partial_- + i\vec{\gamma}_T \cdot \vec{\nabla}_T - m \right) \psi \tag{15.1}$$

where $\partial_\mu = \frac{\partial}{\partial z^\mu}$. But then one notes that the "time" derivative $\partial_+ = \frac{\partial}{\partial z^+}$ acts only on a part of the field ψ, i.e., not all components of ψ are dynamical variables. To see this, one introduces the following decomposition of the field ψ:

$$\begin{aligned} \psi &= \psi_+ + \psi_- \,, \qquad \text{where} \\ \psi_\pm &\equiv P_\pm \psi = \frac{1}{\sqrt{2}} \gamma^0 \gamma^\pm \psi \end{aligned}$$

By using $(\gamma^+)^2 = (\gamma^-)^2 = 0$, we see that these operators $P_\pm$ are projection operators: $P_+ + P_- = 1$, $(P_+)^2 = P_+$, $(P_-)^2 = P_-$, $P_+ P_- = P_- P_+ = 0$. Then we see from (15.1) that the "time derivative" $\partial_+ = \frac{\partial}{\partial z^+}$ does not act on ψ_-, because $\gamma^+ \psi_- = \gamma^+ P_- \psi = \frac{1}{\sqrt{2}} \gamma^+ \gamma^0 \gamma^- \psi = \frac{1}{\sqrt{2}} \left(\gamma^+ \right)^2 \gamma^0 \psi = 0$. The conclusion is that, in the light-cone field theory, only the components

$$\phi \equiv P_+ \psi = \frac{1}{\sqrt{2}} \gamma^0 \gamma^+ \psi \tag{15.2}$$

are dynamical variables, and the remaining part $(P_- \psi)$ should be eliminated by using the field equations, before imposing the quantization conditions.

<u>Home work:</u> (i) Show that the Lagrangian (15.1) can be expressed as follows:

$$\mathcal{L} = \sqrt{2} \left(\psi_+^\dagger \, i\partial_+ \psi_+ + \psi_-^\dagger \, i\partial_- \psi_- \right) - \psi_+^\dagger \left(i\vec{\alpha}_T \cdot \vec{\nabla}_T + \gamma^0 m \right) \psi_- - \psi_-^\dagger \left(i\vec{\alpha}_T \cdot \vec{\nabla}_T + \gamma^0 m \right) \psi_+$$

(ii) Derive the equations of motion (Dirac equations) for ψ_+ and ψ_-.
(iii) Eliminate ψ_-, i.e., express ψ_- in terms of ψ_+. To do this, use the following relation for the inverse of the operator ∂_-:

$$\left(\frac{1}{\partial_-} f \right) (x^-) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}y^- \, \epsilon(x^- - y^-) \, f(y^-)$$

where $\epsilon(x) \equiv \theta(x) - \theta(-x)$.

Returning now to (14.18) and (14.21), we note that

$$\begin{aligned}\overline{\psi}(0)\,\gamma^+\,\psi(z^-) &= \psi^\dagger(0)\gamma^0\gamma^+\psi(z^-) = \sqrt{2}\phi^\dagger(0)\,\phi(z^-) \\ \overline{\psi}(0)\,\gamma^+\,\gamma_5\,\psi(z^-) &= \psi^\dagger(0)\gamma^0\gamma^+\,\gamma_5\psi(z^-) = \sqrt{2}\phi^\dagger(0)\,\gamma_5\phi(z^-)\end{aligned} \tag{15.3}$$

where we used (15.2) and also $P_+^\dagger = P_+$, $(P_+)^2 = P_+$, and $[P_+, \gamma_5] = 0$. We see that $q_i(x)$ and $\Delta q_i(x)$ can be expressed only by the dynamical field $\phi = P_+\,\psi$. In light cone field theory, we must also change the normalization of the hadron state vector $|p, S\rangle$ according to [8]

$$|p, S\rangle \to \sqrt{\frac{p^+}{E_p}}\,|p, S\rangle \tag{15.4}$$

with $\langle p, S|p, S\rangle = V$ as before. By using (15.3) and (15.4), the expressions (14.18) and (14.21) become

$$q_i(x) = p^+\sqrt{2}\int\frac{\mathrm{d}z^-}{2\pi}\,e^{(p^+x)z^-}\,\langle p, S|\phi_i^\dagger(0)\,\phi_i(z^-)|p, S\rangle \tag{15.5}$$

$$\Delta q_i(x) = p^+\sqrt{2}\int\frac{\mathrm{d}z^-}{2\pi}\,e^{(p^+x)z^-}\,\langle p, S|\phi_i^\dagger(0)\,\gamma_5\phi_i(z^-)|p, S\rangle \tag{15.6}$$

Now we introduce the Fourier expansions of the quark field operators $\phi(z^-, \vec{z}_T, z^+)$ and $\phi^\dagger(z^-, \vec{z}_T, z^+)$ for fixed "time" z^+, in the same way as in the usual second quantization method:

$$\phi(z^-, \vec{z}_T, z^+) = \int\frac{\mathrm{d}k^+\,\mathrm{d}^2k_T}{(2\pi)^{3/2}}\sqrt{\frac{m}{k^+}}\sum_{s=\pm1} b_s(k^+, \vec{k}_T)\,u_+(k^+, \vec{k}_T; s)\,e^{-ik^+z^-}\;e^{i\vec{k}_T\cdot\vec{z}_T} + (\text{antiquark term}) \tag{15.7}$$

$$\phi^\dagger(z^-, \vec{z}_T, z^+) = \int\frac{\mathrm{d}k^+\,\mathrm{d}^2k_T}{(2\pi)^{3/2}}\sqrt{\frac{m}{k^+}}\sum_{s=\pm1} b_s^\dagger(k^+, \vec{k}_T)\,u_+^\dagger(k^+, \vec{k}_T; s)\,e^{ik^+z^-}\;e^{-i\vec{k}_T\cdot\vec{z}_T} + (\text{antiquark term}) \tag{15.8}$$

[8] The reason is as follows: Our state $|p, S\rangle$ in "non-covariant normalization" was defined such that $|p, S\rangle\langle p, S|$ is the residue at the pole ($p_0 = E_p$) of the Feynman propagator. That is, the pole term of the Feynman propagator is given by $\dfrac{|p, S\rangle\langle p, S|}{p_0 - E_p}$. This is expressed in terms of light-cone variables as follows:

$$\begin{aligned}\frac{|p, S\rangle\langle p, S|}{p_0 - E_p} &\overset{p_0\simeq E_p}{\equiv} 2E_p\,\frac{|p, S\rangle\langle p, S|}{p^2 - M^2} \\ &= 2E_p\,\frac{|p, S\rangle\langle p, S|}{2p^+p^- - \vec{p}_T^{\,2} - M^2} = \frac{E_p}{p^+}\frac{|p, S\rangle\langle p, S|}{p^- - \epsilon_p}\end{aligned}$$

where $\epsilon_p = \frac{\vec{p}_T^{\,2}+M^2}{2p^-}$. Because the variable p^- plays the role of "energy" (it is conjugate to the "time" variable x^+), the correctly normalized states in the light-cone theory should be defined by the residues of the Feynman propagator at the pole $p^- = \epsilon_p$. This leads to Eq.(15.4).

where $b_s^\dagger(k^+,\vec{k}_T)$ creates a quark with light-cone momentum components $(k^+,\vec{k}_T)$ and (twice the) spin projection $s=\pm 1$ along the spin direction of the nucleon, $b_s(k^+,\vec{k}_T)$ annihilates a quark, and the spinor u_+ is defined by $u_+\equiv P_+ u$, where u is our usual Dirac spinor (normalized by $\overline{u}u=1$), but expressed by light-cone momentum components. (The explicit form of $u_+(k^+,\vec{k}_T;s)$ is given in the Appendix.)

We now introduce the field expansions (15.7) and (15.8) into Eqs.(15.5) and (15.6). In the calculation, we use the following relation which follows from momentum conservation:

$$\langle p,S|b_s^\dagger(k^{+'},\vec{k}_T')\,b_s(k^+,\vec{k}_T)|p,S\rangle=\delta(k^{+'}-k^+)\,\delta^{(2)}(\vec{k}_T'-\vec{k}_T)\,\frac{(2\pi)^3}{V}\,\langle p,S|b_s^\dagger(k^+,\vec{k}_T)\,b_s(k^+,\vec{k}_T)|p,S\rangle$$

and the following spinor matrix elements, which are derived in the Appendix:

$$u_+^\dagger(k^+,\vec{k}_T;s)\,u_+(k^+,\vec{k}_T;s) = \frac{k^+}{\sqrt{2}m} \tag{15.9}$$

$$u_+^\dagger(k^+,\vec{k}_T;s)\,\gamma_5\,u_+(k^+,\vec{k}_T;s) = \pm\frac{k^+}{\sqrt{2}m} \qquad (\text{for } \ s=\pm 1) \tag{15.10}$$

The results are as follows:

$$q_i(x) = \frac{p^+}{V}\int \mathrm{d}^2k_T\,\langle p,S|b_{i,\uparrow}^\dagger(p^+x,\vec{k}_T)\,b_{i,\uparrow}(p^+x,\vec{k}_T)+b_{im\downarrow}^\dagger(p^+x,\vec{k}_T)\,b_{i,\downarrow}(p^+x,\vec{k}_T)|p,S\rangle \tag{15.11}$$

$$\Delta q_i(x) = \frac{p^+}{V}\int \mathrm{d}^2k_T\,\langle p,S|b_{i,\uparrow}^\dagger(p^+x,\vec{k}_T)\,b_{i,\uparrow}(p^+x,\vec{k}_T)-b_{i,\downarrow}^\dagger(p^+x,\vec{k}_T)\,b_{i,\downarrow}(p^+x,\vec{k}_T)|p,S\rangle \tag{15.12}$$

<u>Homework</u>: Derive Eqs.(15.11) and (15.12) by using the relations (15.5) - (15.10).

Check of the sum rules (14.7) and (14.8): Using $k^+=p^+x$ we get

$$\begin{aligned}\int_0^1 \mathrm{d}x\,q_i(x) &= \frac{1}{V}\int_0^{p^+}\mathrm{d}k^+\int \mathrm{d}^2k_T\langle p,S|b_{i,\uparrow}^\dagger(k^+,\vec{k}_T)\,b_{i,\uparrow}(k^+,\vec{k}_T)+b_{i,\downarrow}^\dagger(k^+,\vec{k}_T)\,b_{i,\downarrow}(k^+,\vec{k}_T)|p,S\rangle\\ &= N_i^\uparrow+N_i^\downarrow\equiv N_i \end{aligned}\tag{15.13}$$

$$\begin{aligned}\int_0^1 \mathrm{d}x\,\Delta q_i(x) &= \frac{1}{V}\int_0^{p^+}\mathrm{d}k^+\int \mathrm{d}^2k_T\langle p,S|b_{i,\uparrow}^\dagger(k^+,\vec{k}_T)\,b_{i,\uparrow}(k^+,\vec{k}_T)-b_{i,\downarrow}^\dagger(k^+,\vec{k}_T)\,b_{i,\downarrow}(k^+,\vec{k}_T)|p,S\rangle\\ &= N_i^\uparrow-N_i^\downarrow\equiv \Delta N_i \end{aligned}\tag{15.14}$$

Here we used the familiar fact that $b_{i,s}^\dagger(k^+,\vec{k}_T)b_{i,s}(k^+,\vec{k}_T)$ is the number density of quarks with flavor i, momentum components $(k^+,\vec{k}_T)$, and spin component parallel to the nucleon spin ($s=\uparrow$) or antiparallel to the nucleon spin ($s=\downarrow$). (Remember that our normalization of the state vector is

$\langle p, S|p, S\rangle = V$.)

The results (15.11) and (15.12) clearly show our previous conjectures:

- $q_i(x)$ is the sum of probability densities to find a quark (flavor i) with spin parallel ($\uparrow$) and anti-parallel ($\downarrow$) to the nucleon spin, with light-cone momentum fraction $x = k^+/p^+$. The integral (15.13) gives the number of (valence) quarks with flavor i in the nucleon, and is therefore called the "number sum rule". In the naive quark model for the proton, we expect $N_u = 2$, $N_d = 1$ ($N_u + N_d = 3$).

- $\Delta q_i(x)$ is the difference of probability densities to find a quark (flavor i) with spin parallel ($\uparrow$) and anti-parallel ($\downarrow$) to the nucleon spin, with light-cone momentum fraction $x = k^+/p^+$. The integral (15.14) gives the contribution of (valence) quarks to the spin of the nucleon, and is therefore called the "spin sum rule". In the naive quark model, we expect $\Delta N_u + \Delta N_d = 1$, i.e., 100% of the nucleon spin comes from the spin of the quarks [9].

16 The spin crisis of the proton (an introduction)

Remember from Sect. 13: The spin-dependent structure function of the proton $g_1^{(p)}$ in the Bjorken limit of the parton model, taking into account only up and down quark contributions, takes the form [10]

$$g_1^{(p)}(x) = \frac{1}{2}\left(\frac{4}{9}\,\Delta u^{(p)}(x) + \frac{1}{9}\,\Delta d^{(p)}(x)\right) + (\text{antiquark contributions}) \tag{16.15}$$

For the neutron, we have

$$\begin{aligned} g_1^{(n)}(x) &= \frac{1}{2}\left(\frac{4}{9}\,\Delta u^{(n)}(x) + \frac{1}{9}\,\Delta d^{(n)}(x)\right) + (\text{antiquark contributions}) \\ &= \frac{1}{2}\left(\frac{4}{9}\,\Delta d^{(p)}(x) + \frac{1}{9}\,\Delta u^{(p)}(x)\right) + (\text{antiquark contributions}) \end{aligned} \tag{16.16}$$

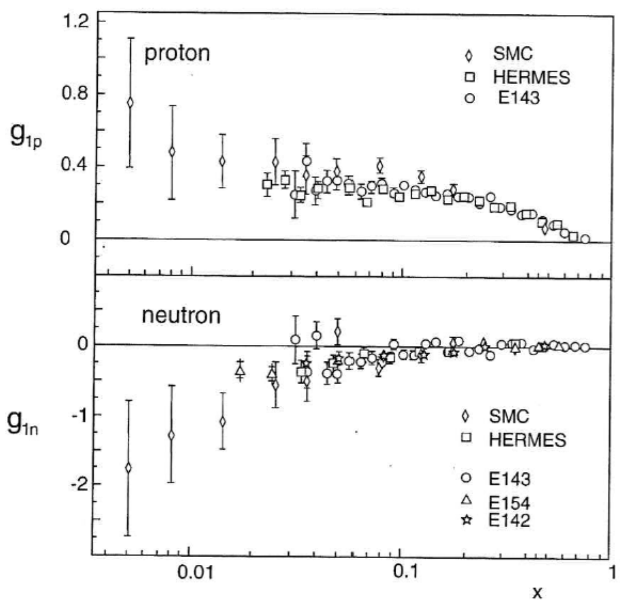
Here we used the flavor symmetry: $\Delta u^{(p)}(x) = \Delta d^{(n)}(x)$, and $\Delta d^{(p)}(x) = \Delta u^{(n)}(x)$.
The parton model predicts that $g_2^{(p)}(x) = g_2^{(n)}(x) = 0$.

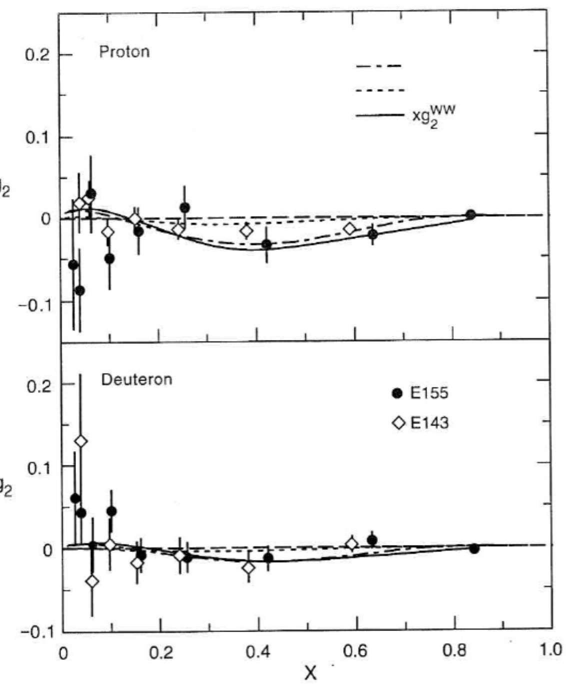
[9] In the SU(6) quark model for the proton, $\Delta N_u = \frac{4}{3}$ and $\Delta N_d = -\frac{1}{3}$, which means that the spins of the two up quarks are parallel to the nucleon spin, and the spin of the down quark is antiparallel to the proton spin.

[10] As in Sect. 12, we simply write $\Delta u^{(p)}(x) \equiv q_u^{(p)}$ for the up-quark distribution in the proton, and $\Delta d^{(p)}(x) \equiv q_d^{(p)}$ for the down-quark distributions.

Experimental data:

As in Sect. 12, we plot experimental data for the structure functions for several values of $Q^2 > 2$ GeV^2 as functions of x:



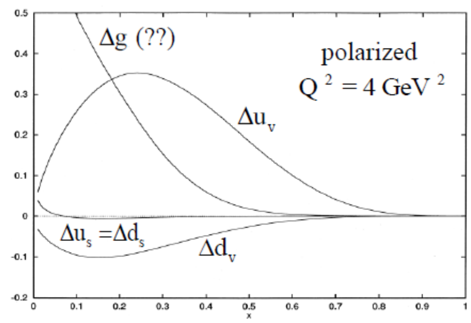


We see that scaling is valid (the structure functions depend only on x, not on Q^2, as long as $Q^2 > 2$ GeV^2), and that g_2 is small (in the parton model it is zero).

From these (and also other) data one can get $\Delta u^{(p)}(x)$ and $\Delta d^{(p)}(x)$. In the extended parton model, these distributions are separated into "valence" (v) quark and "sea" (s) quark contributions: $\Delta u^{(p)}(x) = \Delta u_v^{(p)}(x) + \Delta u_s^{(p)}(x)$. Because the sea quarks always come in $q\overline{q}$ pairs, we have $\Delta u_s^{(p)}(x) = \Delta\overline{u}^{(p)}(x)$, etc. In inclusive electron-proton scattering, only the sum $\Delta u^{(p)}(x) = \Delta u_v^{(p)}(x) + \Delta u_s^{(p)}(x)$ can be measured. But there are other experiments [11], from which one can get the antiquark distributions, like $\Delta\overline{u}^{(p)}(x) = \Delta u_s^{(p)}(x)$, etc. Then the valence quark distributions can be obtained from $\Delta u_v^{(p)}(x) = \Delta u^{(p)}(x) - \Delta u_s^{(p)}(x)$.

[11] These are proton-proton collisions, where a lepton pair ($\ell^+\ell^-$, called "Drell-Yan pair") is observed in the final state. In the parton model, the elementary process is: $(\text{quark})_{\text{hadron}\,1} + (\text{antiquark})_{\text{hadron}\,1} \longrightarrow \gamma \longrightarrow \ell^+\ell^-$. A parton model analysis of this process give information on the product of quark and antiquark distribution functions.

The result of such an analysis of spin dependent parton distributions in the proton looks as follows:



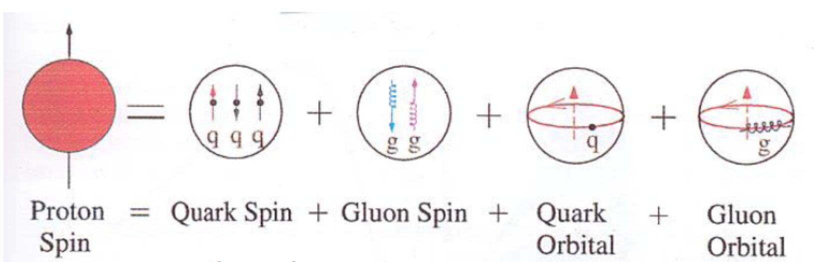
These results are very surprising, because the contribution of the spin of the quarks to the spin of the proton comes out very small: Including the contribution of strange (s) quarks [12], the result is:

$$\Delta u^{(p)} + \Delta d^{(p)} + \Delta s^{(p)} = 0.29 \pm 0.06 \tag{16.17}$$

Here we defined

$$\begin{aligned} \Delta q^{(p)} &\equiv \int_0^1 \mathrm{d}x \, \left(\Delta q^{(p)}(x) + \Delta \overline{q}^{(p)}(x)\right) \\ &= \int_0^1 \mathrm{d}x \, \left(\Delta q_v^{(p)}(x) + 2\Delta q_s^{(p)}(x)\right) \qquad (q = u, d, s) \end{aligned} \tag{16.18}$$

Therefore, only about 30 % of the proton spin is carried by the spin of the quarks and antiquarks! This is called the "spin crisis" (or "spin puzzle"): From where comes the rest of about 70 % ?



(from ref. 4, p. 33)

So, from where comes the surprising result (16.17)? There are 3 necessary informations:

1. The integral of the measured structure function (16.15): Including the s quark contribution,

$$\int_0^1 \mathrm{d}x \, g_1^{(p)}(x) = \frac{1}{2}\left(\frac{4}{9}\Delta u^{(p)} + \frac{1}{9}\Delta d^{(p)} + \frac{1}{9}\Delta s^{(p)}\right) = 0.153 \tag{16.19}$$

[12] The strange quark (s) in the proton is always a "sea quark", i.e., in our notation $\Delta s^{(p)}(x) = \Delta s_s^{(p)} = \Delta \overline{s}^{(p)}$, where the *subscript* s means "sea quark".

2. The axial vector coupling constant ($g_A = G_A/G_V = 1.26$) for the β-decay $n \to p + e^- + \overline{\nu}_e$ (on the quark level: $d \to u + e^- + \overline{\nu}_e$) is defined by the following matrix element of the axial vector current:

$$\begin{aligned}&\langle p, S| \left(\overline{\psi}_u(0)\gamma^\mu\gamma_5\,\psi_u(0) - \overline{\psi}_d(0)\gamma^\mu\gamma_5\,\psi_d(0)\right)|p, S\rangle \\ &\equiv g_A \left(\frac{M}{E_p}\overline{u}(\vec{p}, S)\gamma^\mu\gamma_5\,u(\vec{p}, S)\right) = g_A \frac{M}{E_p} S^\mu \end{aligned} \tag{16.20}$$

Comparison with Eq.(14.6) gives [13]

$$\Delta u^{(p)} - \Delta d^{(p)} = g_A = 1.26 \tag{16.21}$$

This is called the "Bjorken sum rule".

3. In a similar way, the axial vector coupling constant for the β-decay $\Sigma^- \to n + e^- + \overline{\nu}_e$ (on the quark level: $s \to u + e^- + \overline{\nu}_e$) gives the following sum rule:

$$\Delta u^{(p)} + \Delta d^{(p)} - 2\Delta s^{(p)} = 0.59 \tag{16.22}$$

Then from the three independent relations (16.19), (16.21), (16.22), one obtains the surprisingly small value (16.17).

At present, it is not yet clear whether the remaining 70% of the proton spin comes from

- orbital angular momentum of quarks
- spin of gluons
- orbital angular momentum of gluons.

The new <u>electron-ion collider</u> (under construction at Brookhaven Lab in the US) will probably solve this puzzle.

[13] Remember that here we write Δq (with $q = u, d, s$) instead of ΔN_q.

Appendix

The form of the positive energy Dirac spinor with normalization $\overline{u}u$ was given in Sect. 4 of the spring semester (see Eq.(4.16)):

$$u(\vec{k},s) = \sqrt{\frac{E_k+m}{2m}} \begin{pmatrix} \varphi_s \\ \frac{\vec{\sigma}\cdot\vec{k}}{E_k+m}\,\varphi_s \end{pmatrix} \tag{16.23}$$

where φ_s is a 2-component Pauli spinor. In order to get the spinor $u_+ = P_+\,u$, we need to multiply

$$P_+ = \frac{1}{\sqrt{2}}\,\gamma^0\,\gamma^+ = \frac{1}{2}\gamma^0\left(\gamma^0+\gamma^3\right) = \frac{1}{2}\left(1+\alpha^3\right) = \frac{1}{2}\begin{pmatrix} 1 & \sigma_3 \\ \sigma_3 & 1 \end{pmatrix} \tag{16.24}$$

This gives

$$u_+(\vec{k},s) = P_+\,u(\vec{k},s) = \frac{1}{2}\sqrt{\frac{E_k+m}{2m}} \begin{pmatrix} 1+\frac{\sigma_3(\vec{\sigma}\cdot\vec{k})}{E_k+m} \\ \sigma_3+\frac{(\vec{\sigma}\cdot\vec{k})}{E_k+m} \end{pmatrix} \varphi_s \tag{16.25}$$

We can simplify this by writing it in the following form:

$$u_+(\vec{k},s) = \sqrt{\frac{E_k+k^3}{4m}} \begin{pmatrix} U \\ \sigma_3 U \end{pmatrix} \varphi_s \tag{16.26}$$

where we introduced the following unitary 2×2 matrix:

$$U = \sqrt{\frac{E_k+m}{2(E_k+k^3)}}\left(1+\frac{\sigma_3(\vec{\sigma}\cdot\vec{k})}{E_k+m}\right) \tag{16.27}$$

Homework: Confirm the relations (16.25), (16.26), and $U^\dagger U = 1$ (unitarity).

We can now define the following "rotated" 2-component Pauli spinor [14] :

$$\chi_s \equiv U\,\varphi_s \tag{16.28}$$

to express Eq.(16.26) in the simple form

$$u_+(\vec{k},s) = \sqrt{\frac{E_k+k^3}{4m}} \begin{pmatrix} 1 \\ \sigma_3 \end{pmatrix} \chi_s \tag{16.29}$$

Homework: Noting that $E_k + k^3 = k^+\sqrt{2}$, use the spinor (16.29) and the form of the matrix $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to confirm the relations (15.9) and (15.10).

[14] The rotation (16.28) is called a "Melosh rotation".